SELF SIMILAR SOLUTIONS IN ONE-DIMENSIONAL KINETIC MODELS: A PROBABILISTIC VIEW.

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Abstract. This paper deals with a class of Boltzmann equations on the real line, extensions of the well-known Kac caricature. A distinguishing feature of the corresponding equations is that the therein collision gain operators are defined by N-linear smoothing transformations. This kind of problems have been studied, from an essentially analytic viewpoint, in a recent paper by Bobylev, Gamba and Cercignani [5]. Instead, the present work rests exclusively on probabilistic methods, based on techniques pertaining to the classical central limit problem and to the so–called fixed–point equations for probability distributions. An advantage of resorting to methods from the probability theory is that the same results – relative to self–similar solutions – as those obtained by Bobylev, Gamba and Cercignani, are here deduced under weaker conditions. In particular, it is shown how convergence to self–similar solution depends on the belonging of the initial datum to the domain of attraction of a specific stable distribution. Moreover, some results on the speed of convergence are given in terms of Kantorovich-Wasserstein and Zolotarev distances between probability measures.

Keywords: Central limit theorem, Domain of normal attraction, Stable law Kac model, Smoothing transformations, Marked recursive N-ary random trees, Self-similar solution.

Mathematical Subject Classification: 60F05 and 82C40.

1. Introduction

In this paper we consider a kinetic-type evolution equation, introduced and studied in [5], which includes some well-known one dimensional Maxwell models. If $\phi(t,\xi)$ denotes the Fourier-Stieltjes transform

$$\phi(t,\xi) := \int_{\mathbb{R}} e^{i\xi v} \rho_t(dv) \qquad (\xi \in \mathbb{R})$$

of a time dependent probability measure ρ_t on the real line $\mathbb{R}$, the equation under interest is

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(t,\xi) + \phi(t,\xi) = \widehat{Q}\Big(\phi(t,\cdot),\dots,\phi(t,\cdot)\Big)(\xi) & (t>0, \xi \in \mathbb{R}) \\ \phi(0,\xi) = \phi_0(\xi) \end{cases} \tag{1}$$

where, given N characteristic functions $\phi_1,\dots,\phi_N$,

$$\widehat{Q}(\phi_1,\dots,\phi_N)(\xi) := \mathbb{E}\big[\phi_1(A_1\xi)\cdots\phi_N(A_N\xi)\big] \qquad (\xi \in \mathbb{R}). \tag{2}$$

The expectation $\mathbb{E}$ in (2) is taken with respect to the distribution of a given vector $A = (A_1,\dots,A_N)$ of real-valued random variables defined on a probability space $(\Omega,\mathcal{F},P)$. The initial condition ϕ_0

Date: March 30, 2010.

is a characteristic function of a prescribed real random variable X_0 with distribution function $F_0(x)$.

Notice that different equations for probability dynamics considered in literature are special cases of (1): the one dimensional Kac caricature [15], some one dimensional dissipative Maxwell models [3, 21, 23], some mean conservative models used to describe economical dynamics see, e.g. [20, 22], some models for mixture of Maxwell gases [6].

For simplicity of notation in the rest of the paper we write $\hat{Q}(\phi)$ instead of $\hat{Q}(\phi, \dots, \phi)$.

The aim of this paper is to study the asymptotic behavior of the solution ϕ of (1) as $t \to +\infty$. One can distinguish two different situations:

- the solution $\phi(t,\xi)$ converges, as $t \to +\infty$, to a stationary solution, i.e. a characteristic function ϕ_∞ such that

$$\phi_\infty = \hat{Q}(\phi_\infty); \tag{3}$$

- there exists μ^* (depending on the initial condition ϕ_0) such that the rescaled solution

$$w(t,\xi) := \phi(t, e^{-\mu^* t}\xi) \tag{4}$$

  converges as $t \to +\infty$ to a non degenerate limit.

To understand the nature of this limit let us observe that the re-scaled solution, w, satisfies the following new equation

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,\xi) + \mu^* \xi \dfrac{\partial}{\partial \xi} w(t,\xi) + w(t,\xi) = \widehat{Q}(w(t,\cdot))(\xi) \qquad (t > 0, \xi \in \mathbb{R}) \\ w(0,\xi) = \phi_0(\xi). \end{cases} \tag{5}$$

When $\mu^* = 0$ equation (5) reduces to (1) and, clearly, w is simply ϕ. The stationary equation associated to (5) is, for every μ^*,

$$\mu^* \xi \frac{\partial}{\partial \xi} w_\infty(\xi) + w_\infty(\xi) = \widehat{Q}(w_\infty)(\xi)$$

which can be re-written, after easy computations, as an integral equation for a Fourier-Stieltjes transform

$$w_\infty(\xi) = \int_0^1 \widehat{Q}(w_\infty)(\tau^{\mu^*}\xi) d\tau. \tag{6}$$

It is important to note that, if a characteristic function w_∞ satisfies (6), then

$$\phi(t,\xi) := w_\infty\big(\exp\{\mu^* t\}\xi\big)$$

satisfies the original Kac-like equation (1) with initial condition $\phi_0(\xi) = w_\infty(\xi)$. Following [5], we shall use the name self-similar solution for a solution w_∞ of (6) (when it exists), although the name self-similar solution is usually devoted to $w_\infty(\exp\{\mu^* t\}\xi)$.

In term of random variables, (6) becomes

$$X \stackrel{\mathcal{L}}{=} \Theta^{\mu^*} \sum_{i=1}^{N} A_i X_i \tag{7}$$

where $(X, X_1, \ldots, X_N)$ are stochastically independent random variables with the same characteristic function w_∞, Θ is a random variable with uniform distribution on $(0,1)$ and $(X, X_1, \ldots, X_N)$, Θ and $(A_1, \ldots, A_N)$ are stochastically independent. Moreover, $Z_1 \stackrel{\mathcal{L}}{=} Z_2$ means that the random variables Z_1 and Z_2 have the same law. $\hat{Q}$ is usually called smoothing transformation and equations of kind (6)-(7) are referred to as fixed point equations for distributions.

In [5] Maxwell models of type (1) are considered from a very general point of view and some key properties that lead to the self-similar asymptotics are established mainly by analytic techniques. The goal of our paper is to study convergence to self-similar solutions by means of probabilistic methods. Via a suitable probabilistic representation of the solution of (1) we resort to *central limit theorems* and *fixed point equations for distributions*. In this way we are able to extend some results presented in [5]. The main result we obtained is the proof of long-time convergence of the rescaled solution to a self-similar solution essentially under the natural hypothesis that the initial condition belongs to the domain of normal attraction of a stable distribution. Our approach is a generalization, to the study of convergence to self–similar solutions, of the one developed in [2] to study the convergence to stationary solutions for the problem (1)-(2) with $N = 2$.

The general idea to represent solutions to Kac-like equations in a probabilistic way dates back at least to [17]; this approach has been fully formalized and employed in the derivation of various results in the last decade, see e.g.[7, 12]. For the original Kac equation, probabilistic methods have been used in many papers, see [24] for a review.

The paper is organized as follows. Section 2 contains the the statements of our main theorems. In Section 3 we derive the stochastic representation of solutions to (1). Section 4 contains the statements of some intermediate results concerning sums of random variables indexed by random N-ary recursive trees. All proofs are completed in Section 5.

2. Main results

From now on we assume that A_i are non-negative random variables such that

$$\begin{aligned} &P\left\{\sum_{i=1}^{N} \mathbb{I}\{A_i > 0\} \in \{0,1\}\right\} < 1, \quad \mathbb{E}\left[\sum_{i=1}^{N} \mathbb{I}\{A_i > 0\}\right] > 1, \\ &P\{A_i = 0 \text{ or } 1, \forall\, i = 1, \ldots, N\} < 1. \end{aligned} \tag{8}$$

In the theorems below the initial condition F_0 will satisfy one of the following hypotheses $(\mathbf{H}_\gamma)$, where γ belongs to $(0,2]$.

$(\mathbf{H}_1)$

either (a) $\int_{\mathbb{R}} |v| dF_0(v) < +\infty$ *and* $m_0 = \int_{\mathbb{R}} v dF_0(v)$

or (b) F_0 *is a symmetric distribution function and satisfies the condition*

$$\lim_{x\to+\infty} x(1-F_0(x)) = c_0^+ < +\infty, \qquad \lim_{x\to-\infty} |x| F_0(x) = c_0^+ < +\infty \tag{9}$$

with $c_0^+ > 0$*;*

$(\mathbf{H}_2)$ $0 < \sigma_0^2 := \int_{\mathbb{R}} |v|^2 dF_0(v) < +\infty$ *and* $\int_{\mathbb{R}} v dF_0(v) = 0$*;*

if $\gamma \in (0,1) \cup (1,2)$

$(\mathbf{H}_\gamma)$ F_0 *satisfies the condition*

$$\lim_{x\to+\infty} x^\gamma(1-F_0(x)) = c_0^+ < +\infty, \qquad \lim_{x\to-\infty} |x|^\gamma F_0(x) = c_0^- < +\infty \tag{10}$$

with $c_0^+ + c_0^- > 0$ *and, in addition,* $\int_{\mathbb{R}} v dF_0(v) = 0$ *if* $\gamma > 1$.

Accordingly, define

$$\hat{g}_\gamma(\xi) := \begin{cases} \exp\{i m_0 \xi\} & \text{if } \gamma = 1 \text{ and (a) of } (\mathbf{H}_1) \text{ holds,} \\ \exp\{-\pi c_0^+ |\xi|\} & \text{if } \gamma = 1 \text{ and (b) of } (\mathbf{H}_1) \text{ holds,} \\ \exp\{-\sigma_0^2 |\xi|^2/2\} & \text{if } \gamma = 2 \text{ and } (\mathbf{H}_2) \text{ holds} \\ \exp\{-k_0 |\xi|^\gamma (1 - i\eta_0 \tan(\pi\gamma/2) \operatorname{sign} \xi)\} & \text{if } \gamma \in (0,1)\cup(1,2) \text{ and } (\mathbf{H}_\gamma) \text{ holds,} \end{cases} \tag{11}$$

where

$$k_0 = (c_0^+ + c_0^-) \frac{\pi}{2\Gamma(\gamma)\sin(\pi\gamma/2)}, \qquad \eta_0 = \frac{c_0^+ - c_0^-}{c_0^+ + c_0^-}. \tag{12}$$

Notice that, except when $\gamma = 1$ and (a) of $(\mathbf{H}_1)$ holds, $\hat{g}_\gamma$ is the Fourier-Stieltjes transform of *a centered stable law* of exponent γ and (10) of $(\mathbf{H}_\gamma)$ is equivalent to say that F_0 belongs to the domain of normal attraction of a γ-stable law g_γ with Fourier-Stieltjes transform $\hat{g}_\gamma$. See, for example, Chapter 17 of [11].

It is worthwhile to recall that a distribution function F_0 belongs to the domain of normal attraction of a stable law of exponent γ if for any sequence of independent and identically distributed real-valued random variables $(X_n)_{n\geq 1}$ with common distribution function F_0, there exists a sequence of real numbers $(c_n)_{n\geq1}$ such that the law of $n^{-1/\gamma}\sum_{i=1}^n X_i - c_n$ converges weakly to a stable law of exponent $\gamma \in (0,2]$.

2.1. **Convergence to self-similar solutions.** Our main result states that, under suitable assumptions, the rescaled solution w, defined in (4), converges to a mixture of centered stable characteristic functions. The Fourier-Stieltjes transform of the mixing measure will be characterized as a particular solution of the integral equation

$$v(\xi) = \int_0^1 \mathbb{E}\left[\prod_{i=1}^N v(\xi A_i^\gamma \tau^{\mathcal{S}(\gamma)})\right] d\tau \tag{13}$$

where $\mathcal{S} : [0,\infty) \to [-1,\infty]$ is the convex function defined by

$$\mathcal{S}(s) = \mathbb{E}\left[\sum_{j=1}^{N} A_j^s\right] - 1, \tag{14}$$

with the convention that $0^0 = 0$. Note that, thanks to (8), one has $0 < \mathcal{S}(0) \leq N-1$, hence if $\mathcal{S}(s) < +\infty$ for some s then $\mathcal{S}(q) < +\infty$ for every q in $(0,s)$. We point out that there is a simple connection between the function $s \mapsto \mathcal{S}(s)$, widely used in the probabilistic fixed point literature, and the so–called spectral function $s \mapsto \mu(s)$ introduced in [5], more exactly

$$\mu(s) := \frac{\mathcal{S}(s)}{s} \qquad (s > 0).$$

We collect in the next proposition some useful results concerning equation (13).

Proposition 2.1. *Fix γ in $(0,2]$. Assume that $\mu(\delta) < \mu(\gamma) < +\infty$ for some $\delta > \gamma$. Then*

(i) *there is a unique probability distribution $\zeta_{\infty,\gamma}$ on $\mathbb{R}^+$ such that $v_{\infty,\gamma}(\xi) = \int_{\mathbb{R}} e^{i\xi z}\zeta_{\infty,\gamma}(dz)$ is a solution of (13) with $\int_{\mathbb{R}^+} z\zeta_{\infty,\gamma}(dz) = 1$;*

(ii) *the equation $\mu(q) - \mu(\gamma) = 0$ has at most one solution $q_\gamma^* \neq \gamma$, and we set, by convention, $q_\gamma^* := +\infty$ if the unique solution is $q = \gamma$;*

(iii) *$\zeta_{\infty,\gamma}$ is degenerate if and only $\sum_{i=1}^N A_i^\gamma = 1$ almost surely. Moreover, if $P\{\sum_{i=1}^N A_i^\gamma = 1\} < 1$ and $p > \gamma$, $\int_{\mathbb{R}^+} z^{\frac{p}{\gamma}}\zeta_{\infty,\gamma}(dz) < +\infty$ if and only if $p < q_\gamma^*$.*

In the next theorems we assume that $(\mathbf{H}_\gamma)$ holds true for some γ in $(0,2]$ and we study the self-similar limit of the rescaled solution w for $\mu^* = \mu(\gamma)$. We will see that the non-degeneracy of the limit will depend on the shape of the spectral function μ.

Theorem 2.2. *Let (8) be in force. Assume that $(\mathbf{H}_\gamma)$ holds true for some γ in $(0,2]$ and that $\mu(\delta) < \mu(\gamma) < +\infty$, for some $\delta > \gamma$. Then, there is a probability measure $\rho_{\infty,\gamma}$ such that:*

(i) *the characteristic function of $\rho_{\infty,\gamma}$ is a self-similar solution, i.e. $w_{\infty,\gamma}(\xi) := \int_{\mathbb{R}} e^{i\xi v}\rho_{\infty,\gamma}(dv)$ is a solution of (6) for $\mu^* = \mu(\gamma)$ and*

$$\lim_{t\to+\infty} \phi(t, e^{-t\mu(\gamma)}\xi) = w_{\infty,\gamma}(\xi) \tag{15}$$

for every $\xi \in \mathbb{R}$. Moreover,

$$w_{\infty,\gamma}(\xi) = \int_{\mathbb{R}^+} \hat{g}_\gamma(\xi z^{\frac{1}{\gamma}})\zeta_{\infty,\gamma}(dz)$$

where $\zeta_{\infty,\gamma}$ is given in (i) *of* Proposition 2.1 *and $\hat{g}_\gamma$ is defined in* (11).

(ii) *If $\gamma \neq 1, 2$ or if $\gamma = 1$ and* (b) *of $(\mathbf{H}_1)$ holds, then $\rho_{\infty,\gamma}$ is a γ-stable distribution if and only if $\sum_{i=1}^N A_i^\gamma = 1$ almost surely. Moreover $\int_{\mathbb{R}} |v|^p\rho_{\infty,\gamma}(dv) < +\infty$ if and only if $p < \gamma$.*

(iii) *If $\gamma = 1$ and* (a) *of $(\mathbf{H}_1)$ holds, then $\rho_{\infty,\gamma} = \delta_{m_0}$ if and only if $\sum_{i=1}^N A_i = 1$ almost surely. Moreover, if $P\{\sum_{i=1}^N A_i = 1\} < 1$, then $\int_{\mathbb{R}} |v|^p\rho_{\infty,1}(dv) < +\infty$ for $p > 1$ if and only if $p < q_1^*$ (where q_1^* is defined in* (ii) *of* Proposition 2.1).

(iv) *If $\gamma = 2$, then $\rho_{\infty,2}$ is a gaussian distribution if and only if $\sum_{i=1}^N A_i^2 = 1$ almost surely. Moreover, if $P\{\sum_{i=1}^N A_i^2 = 1\} < 1$, then $\int_{\mathbb{R}} |v|^p \rho_{\infty,2}(dv) < +\infty$ for $p > 2$ if and only if $p < q_2^*$ (where q_2^* is defined in* (ii) *of* Proposition 2.1).

The following theorem considers the cases in which the rescaling $e^{-\mu(\gamma)t}$ provides a degenerate limiting solution.

Theorem 2.3. *Let* (8) *be in force. Assume that* ($\mathbf{H}_\gamma$) *holds true for some γ in $(0, 2]$. If $\mu(\delta) < \mu(\gamma) < +\infty$, for some $0 < \delta < \gamma$, then*

$$\lim_{t\to+\infty} \phi(t, e^{-t\mu(\gamma)}\xi) = 1 \qquad (\xi \in \mathbb{R}).$$

2.2. **Comparison with previous results.** In [5] the Chauchy problem (1)–(2) is studied under the hypothesis that $A_1, \dots, A_N$ are exchangeable random variables with finite moments of any order. The convergence of the rescaled solution to the self-similar one is obtained under the same hypotheses on the spectral function μ assumed in Theorem 2.2, in two different situations:

- when F_0 is a symmetric distribution function and

$$1 - \phi_0(\xi) = |\xi|^\gamma + O(|\xi|^{\gamma+\epsilon}) \qquad (\xi \to 0) \tag{16}$$

for some $\gamma \le 2$ and $\epsilon > 0$;

- when F_0 is supported by $\mathbb{R}^+$ and its Laplace transfom $L_0(\xi) = \int_{\mathbb{R}^+} e^{-\xi v} dF_0(v)$ satisfies

$$1 - L_0(\xi) = \xi^\gamma + O(|\xi|^{\gamma+\epsilon}) \qquad (\xi \to 0^+) \tag{17}$$

for some $\gamma \le 1$ and $\epsilon > 0$.

Some results on moments of the self-similar solutions are proved under the stronger assumption that the distributions of the A_i's have compact support.

The probabilistic approach enables us to weaken the hypotheses both on the A_i's and on the initial condition ϕ_0. In particular, we don't require neither the symmetry (except for (b) in assumption ($\mathbf{H}_1$)) nor the positiveness of the initial data. Moreover, ($\mathbf{H}_\gamma$) is weaker than (16)-(17). In point of fact if F_0 is symmetric, then it satisfies (10) for $0 < \gamma < 2$ if and only if

$$1 - \phi_0(\xi) = k_0|\xi|^\gamma(1 + o(1))$$

as $|\xi| \to 0$, and $\sigma_0^2 < +\infty$ if and only if $1 - \phi_0(\xi) = \frac{\sigma_0^2}{2}|\xi|^2(1 + o(1))$. See Théorème 1.3 of [13]. On the other way, if F_0 is supported by $\mathbb{R}^+$ and its Laplace transfom satisfies (17) for $\gamma < 1$, by Theorem 4 in Section XII.5 of [10], it follows that (10) holds true. Finally, if (17) holds for $\gamma = 1$, it follows immediately that $\int_{\mathbb{R}^+} v dF_0(v) < +\infty$.

2.3. **Rates of convergence.** Let V_t be a random variable whose characteristic function is the unique solution $\phi(t,\xi)$ to problem (1). Such a V_t is given explicitly in Section 3 (see Proposition 3.2). Theorem 2.2 states the convergence in distribution of $e^{-\mu(\gamma)t}V_t$ to a random variable V_∞ with probability distribution $\rho_{\infty,\gamma}$. Under some additional hypotheses, if $\gamma \neq 2$, the (exponential) rate at which this convergence takes place can be quantified in suitable Wasserstein metrics as stated in Theorem 2.4 below.

Recall that the Wasserstein distance of order $\delta > 0$ between two random variables X and Y, or equivalently between their probability distributions, is defined by

$$l_\delta(X,Y) := \inf_{(X',Y')} (\mathbb{E}|X'-Y'|^\delta)^{\frac{1}{\max(\delta,1)}}. \tag{18}$$

The infimum is taken over all pairs (X',Y') of real random variables whose marginal distributions are the same as those of X and Y, respectively. In general, the infimum in (18) may be infinite; a sufficient (but not necessary) condition for finite distance is that both $\mathbb{E}|X|^\delta < +\infty$ and $\mathbb{E}|Y|^\delta < +\infty$. For more information on Wasserstein distances see, for example, [25].

The next theorem is the natural generalization of Theorem 5 in [2].

Theorem 2.4. *Let* (8) *be in force. Assume that* $(\mathbf{H}_\gamma)$ *holds true for some* γ *in* $(0,2)$ *and that* $\mu(\delta) < \mu(\gamma)$, *for some* $\gamma < \delta$ *with* $1 \leq \gamma < \delta \leq 2$ *or* $\gamma < \delta \leq 1$. *Let* V_t *and* V_∞ *be as above. Then*

$$l_\delta(e^{-\mu(\gamma)t}V_t, V_\infty)^{\max(\delta,1)} \leq c\, l_\delta(X_0, V_\infty)^{\max(\delta,1)} e^{-t\delta[\mu(\gamma)-\mu(\delta)]}, \tag{19}$$

with $c = 1$ *if* $\delta \leq 1$ *and* $c = 2$ *otherwise.*

Clearly, (19) is meaningful only if $l_\delta(X_0, V_\infty) < +\infty$. When $\gamma = 1$ and (a) of $(\mathbf{H}_1)$ holds, it follows that $\mathbb{E}|V_\infty|^\delta < +\infty$ by (iii) of Theorem 2.2, since it is easy to see that $\delta < q_1^*$. Hence, in this case, $l_\delta(X_0, V_\infty) < +\infty$ whenever $\mathbb{E}|X_0|^\delta < +\infty$. When $\gamma \neq 1$ or when $\gamma = 1$ and (b) of $(\mathbf{H}_1)$ holds, the requirement $l_\delta(X_0, V_\infty) < +\infty$ is non-trivial since, by Theorem 2.2, one has $\mathbb{E}[|V_\infty|^\delta] = +\infty$. The following Lemma provides a sufficient criterion tailored to the situation at hand.

Lemma 2.5. *Assume, in addition to the hypotheses of* Theorem 2.2, *that* $\delta < 2\gamma$ *and that* F_0 *satisfies* $(\mathbf{H}_\gamma)$ *in the more restrictive sense that there exists a constant* $K > 0$ *and some* $0 < \epsilon < 1$ *with*

$$|1 - c_0^+ x^{-\gamma} - F_0(x)| < K x^{-(\gamma+\epsilon)} \quad \text{for } x > 0, \tag{20}$$

$$|F_0(x) - c_0^-(-x)^{-\gamma}| < K(-x)^{-(\gamma+\epsilon)} \quad \text{for } x < 0. \tag{21}$$

Provided that $\delta < \gamma/(1-\epsilon)$, *it follows* $l_\delta(X_0, V_\infty) < +\infty$.

We have not been able to prove Theorem 2.4 for $\gamma = 2$. On the other hand we are able to give the speed of convergence for every γ in $(0, 2]$ with respect to the Zolotarev metrics $\mathcal{Z}_s$. The metric $\mathcal{Z}_s$ is defined, for $s = m + \alpha$, m being a non-negative integer and $0 < \alpha \leq 1$, by

$$\mathcal{Z}_s(X, Y) := \sup\{\mathbb{E}[f(X) - f(Y)] : \ f \in \mathcal{F}_s\}, \tag{22}$$

where $\mathcal{F}_s$ is the set of real valued functions on $\mathbb{R}$ which at all points have the mth derivatives such that $|f^{(m)}(x) - f^{(m)}(y)| \leq |x - y|^\alpha$. For more information see, e.g., [31].

In general the finiteness condition $\mathcal{Z}_s(X, Y)$ is not easy to check. It turns out that if $\int x^r(dF_X(x) - dF_Y(x)) = 0$ for any integer $r \leq m$ and $\int |x|^s |dF_X(x) - dF_Y(x)| < +\infty$, where F_X and F_Y are the distribution functions of X and Y, then $\mathcal{Z}_s(X, Y) < +\infty$. See Theorems 1.5.7 in [31]. The estimates proved in the next theorem are interesting in particular for the case $\gamma = 2$, for which the above sufficient conditions for the finiteness of $\mathcal{Z}_s(X_0, V_\infty)$ are easily verified.

Theorem 2.6. *Let (8) be in force. Assume that $(\mathbf{H}_\gamma)$ holds true for some γ in $(0, 2]$ and that $\mu(\delta) < \mu(\gamma)$, for some $\gamma < \delta$. Let V_t and V_∞ be as above. Then*

$$\mathcal{Z}_\delta(e^{-\mu(\gamma)t} V_t, V_\infty) \leq \mathcal{Z}_\delta(X_0, V_\infty) e^{-t\delta[\mu(\gamma) - \mu(\delta)]}. \tag{23}$$

In particular, if $\gamma = 2$, $\delta \leq 3$ and $\mathbb{E}|X_0|^\delta < +\infty$, then

$$\mathcal{Z}_\delta(e^{-\mu(\gamma)t} V_t, V_\infty) \leq c e^{-t\delta[\mu(\gamma) - \mu(\delta)]} \tag{24}$$

where

$$c := \frac{1}{\Gamma(1 + \delta)} \Big(\mathbb{E}|X_0|^\delta + E|V_\infty|^\delta \Big) < +\infty.$$

3. Marked recursive N-ary random trees and probabilistic interpretation of the solutions

The notion of N-ary random trees will be used to describe, in a probabilistic way, the solution of equation (1). This approach is a generalization of the probabilistic representation presented in [2], where binary trees were considered in order to describe the solution when $\hat{Q}$ is a bilinear smoothing transformation.

3.1. **Random N-ary recursive trees.** Recall that a rooted tree is said to be a planar tree when successors of the root and recursively the successors of each node are equipped with a left-to-right-order. For any integer number $N \geq 2$ an N-ary tree is a (planar and rooted) tree where each node is either a leaf (that is, it has no successor) or it has N successors. We define the size of the N-ary tree t, in symbol $|t|$, by the number of internal nodes. Any N-ary tree with $Nk + 1$ nodes has size k and possesses $f_k := (N - 1)k + 1$ leaves. We now describe a (natural) tree evolution process which gives rise to the so called "random N-ary recursive tree". The evolution process starts with T_0, an empty tree, that is, with just an external node (the root). The first step in the growth

process is to replace this external node by an internal one with N successors that are leaves, in this way we get T_1. Then with probability 1/N (i.e. the number of leaves) one of these N leaves is selected and again replaced by an internal node with N successors. In this way one continues. At every time k, T_k is an N-ary tree with k internal nodes.

A very important issue is that N-ary trees have a recursive structure. More precisely we can use the following recursive definition of N-ary trees: an N-ary tree t is either just an external node or an internal node with N subtrees that are again N-ary trees. We shall denote these subtrees by $t^{(1)},\dots,t^{(N)}$.

Recall also that every N-ary tree can be seen as a subset of

$$\mathbb{U} := \{\emptyset\} \cup [\cup_{k\geq 1}\{1,2,\dots,N\}^k].$$

As usual $\emptyset$ is the root and if $v = (v_1,\dots,v_k)$ ($v_i \in \{1,\dots,N\}$) is a node of an N-ary tree then the length of v is $|v| := k$, moreover $(v, v_{k+1}) := (v_1,\dots,v_k,v_{k+1})$, and $(v,\emptyset) := v$. For every $1 \leq i \leq k$, set $v|i := (v_1,\dots,v_i)$ and $v|0 = \emptyset$. Finally, given an N-ary tree t we shall denote by $\mathcal{L}(t)$ the set of the leaves of t. For more details on N-ary recursive trees see, for instance, [8].

For every integer $k \geq 1$ set

$$\mathcal{J}_k := \{\underline{i} = (i_1,\dots,i_N) \in \mathbb{N}_0^N : \sum_{j=1}^N i_j = k-1\}$$

where $\mathbb{N}_0 = 0 \cup \mathbb{N}$ and denote by $\mathbb{T}_{k,N}$ the set of all N-ary trees with size k. Notice that $\underline{i} \in \mathcal{J}_1$ if, and only if, $i_1 = \dots = i_N = 0$. Finally, for every $\underline{i}$ in $\mathcal{J}_k$ set

$$C^k_{\underline{i}} = \{t \in \mathbb{T}_{k,N} : |t^{(j)}| = i_j,\ j = 1,\dots,N\}.$$

The following Proposition states some properties of random N-ary recursive trees.

Proposition 3.1. *Let $(T_k)_{k\geq 1}$ be a sequence of random N-ary recursive trees. For every $k \geq 1$, every $\underline{i}$ in $\mathcal{J}_k$ and every t in $\mathbb{T}_{k,N}$,*

$$P\{T^{(1)}_k = t^{(1)},\dots,T^{(N)}_k = t^{(N)} | T_k \in C^k_{\underline{i}}\} = \prod_{j=1}^N P\{T_{|t^{(j)}|} = t^{(j)}\}\mathbb{I}\{|t^{(j)}| = i_j\} \tag{25}$$

and

$$P\{T_k \in C^k_{\underline{i}}\} = p_k(\underline{i}) \tag{26}$$

where for $k \geq 1$,

$$p_k(\underline{i}) := \binom{k-1}{i_1,\dots,i_N} \frac{\prod_{l=1}^N \prod_{m=0}^{i_l-1} f_m}{\prod_{r=0}^{k-1} f_r}, \tag{27}$$

with the convention that $\prod_{r=0}^{-1} f_r = 1$. Finally, when $n \to +\infty$, $(|T^{(1)}_n|/n,\dots,|T^{(N)}_n|/n)$ converges almost surely to a vector $(U_1,\dots,U_N)$ with Dirichlet distribution of parameters $(1/(N-1),\dots,1/(N-1))$.

3.2. **Wild series and probabilistic representation of the solutions.** To start with we will write the Wild series expansion of $\phi(\cdot,t)$. Such kind of expansion can be easily derived using a general result contained in [16]. For every $t \geq 0$ and $k \in \mathbb{N}_0$, set

$$b_k := \frac{\prod_{i=0}^{k-1} f_i}{(N-1)^k k!}$$

and

$$\zeta(t,k) := b_k e^{-t}\Big(1-e^{-(N-1)t}\Big)^k. \tag{28}$$

Remark 1. *Note that $\zeta(t,\cdot)$ is the probability density of a Negative-Binomial random variable of parameters $(1/(N-1), e^{-(N-1)t})$. Indeed, since $f_i = (N-1)i+1$ for $i = 0,1,\dots$*

$$b_k = \frac{(1/(N-1))_k}{k!}$$

where for every non-negative real number r and every non-negative integer n

$$(r)_n = \prod_{i=0}^{n-1}(r+i) = \frac{\Gamma(r+n)}{\Gamma(r)}.$$

and $(r)_0 = 1$.

Using Remark 1 above and Theorem 1 in [16] it is a simple matter to deduce that the unique global solution to (1) is given by

$$\phi(t,\xi) = \sum_{k\geq 0}\zeta(t,k)q_k(\xi)$$

where $(q_k)_k$ is a sequence of characteristic functions recursively defined by setting $q_0(\xi) = \phi_0(\xi)$ and, for $k \geq 1$,

$$q_k(\xi) = \sum_{\underline{i}\in\mathcal{J}_k} p_k(\underline{i})\widehat{Q}(q_{i_1},\dots,q_{i_N})(\xi)$$

where p_k is defined in (27). This representation is the generalization of the Wild series, which is obtained, when $N=2$, in [29]. It is easy to see that $\phi(t,\cdot)$ is a characteristic function.

The Wild series expansion suggests a probabilistic interpretation for the solutions as sums of random variables indexed by N-ary recursive random trees. On a sufficiently large probability space $(\Omega,\mathcal{F},P)$, let the following be given:

- a family $(X_v)_{v\in\mathbb{U}}$ of independent and identically distributed random variables with common distribution function F_0;
- a family $\big(A_1(v),A_2(v),\dots,A_N(v)\big)_{v\in\mathbb{U}}$ of independent and identically distributed positive random vectors with the same distribution of $(A_1,\dots,A_N)$;
- a sequence of N-ary recursive random trees $(T_n)_{n\in\mathbb{N}}$;
- a stochastic process $(\nu_t)_{t\geq 0}$ with values in $\mathbb{N}_0$ such that $P\{\nu_t = k\} = \zeta(t,k)$ for every integer $k \geq 0$, where $\zeta(t,k)$ is defined in (28).

Write $A(v) = (A_1(v), A_2(v), \dots, A_N(v))$ and assume further that

$$(A(v))_{v\in\mathbb{U}}, \quad (T_n)_{n\geq 1}, \quad (X_v)_{v\in\mathbb{U}} \quad \text{and} \quad (\nu_t)_{t>0}$$

are stochastically independent.

For each node $v = (v_1, \dots, v_k)$ in $\mathbb{U}$ set

$$\varpi(v) := \prod_{i=0}^{|v|-1} A_{v_{i+1}}(v|i)$$

and $\varpi(\emptyset) = 1$. Now recall that $\mathcal{L}(T_n)$ is the set of leaves of T_n and define

$$W_0 := X_\emptyset$$

and, for every $n \geq 1$,

$$W_n := \sum_{v\in\mathcal{L}(T_n)} \varpi(v) X_v. \tag{29}$$

Proposition 3.2. *Equation* (1) *has a unique solution* $\phi(t,\cdot)$*, which coincides with the characteristic function of* $V_t := W_{\nu_t}$*.*

Let us conclude this section rewriting W_n in an alternative form. In the following we will use both forms, according to our convenience. For each $n \geq 1$ we shall denote by

$$\{\beta_{1,n}, \dots, \beta_{f_n,n}\}$$

the weights associated to the leaves of T_n, that is if

$$\mathcal{L}(T_n) = \{L_{1,n}, \dots, L_{f_n,n}\}$$

(in left-right order) $\beta_{i,n} = \varpi(L_{i,n})$. Hence we can rewrite W_n as

$$W_n = \sum_{j=1}^{f_n} \beta_{j,n} X_{j,n}$$

where $X_{j,n} := X_{L_{j,n}}$.

4. Some limit theorems for sums of random variables indexed by N-ary recursive trees.

Let us sketch our approach to the study of the asymptotic behavior of $\phi(t, e^{-\mu(\gamma)t}\xi)$. From the probabilistic interpretation we obtain that $\phi(t, e^{-\mu(\gamma)t}\xi)$ is the characteristic function of the rescaled random variable $e^{-\mu(\gamma)t}V_t = e^{-\mu(\gamma)t}W_{\nu_t}$. Hence, we look for a positive function $n \mapsto m_n(\gamma)$ such that

$$(N_t(\gamma), \tilde{W}_{\nu_t}) := \left(e^{-\mu(\gamma)t} m_{\nu_t}(\gamma)^{\frac{1}{\gamma}}, \sum_{j=1}^{f_{\nu_t}} \frac{\beta_{j,\nu_t}}{m_{\nu_t}(\gamma)^{\frac{1}{\gamma}}} X_{j,\nu_t} \right) \tag{30}$$

converges weakly as $t \to +\infty$, in order to obtain the convergence of $e^{-\mu(\gamma)t} V_t = N_t(\gamma) \tilde{W}_{\nu_t}$. This will be done in several steps. First of all we will study, for suitable $m_n(\gamma)$'s, the weak limit of

$$\tilde{W}_n := \frac{W_n}{m_n(\gamma)^{\frac{1}{\gamma}}} = \frac{1}{m_n(\gamma)^{\frac{1}{\gamma}}} \sum_{j=1}^{f_n} \beta_{j,n} X_{j,n}, \tag{31}$$

which is a sum of random variables from a triangular array. Notice that a direct application of a central limit theorem is not possible, since the weights $m_n(\gamma)^{-\frac{1}{\gamma}} \beta_{j,n}$ are not independent. However, one can apply a central limit theorem to the conditional law of $\tilde{W}_n$, given the array of weights $\beta_{j,n}$ and $(T_n)_{n\geq 1}$. To this end, we shall prove that under suitable assumptions, if

$$m_n(\gamma) := \prod_{k=0}^{n-1} \Big(1 + \frac{\mathcal{S}(\gamma)}{f_k}\Big), \tag{32}$$

$\mathcal{S}(\gamma)$ being defined in (14), then

$$\tilde{M}_n(\gamma) := \frac{1}{m_n(\gamma)} \sum_{j=1}^{f_n} \beta_{j,n}^{\gamma}$$

converges a.s. to a limit $\tilde{M}_\infty(\gamma)$ and that $\max_{j=1,\dots,f_n} \beta_{j,n} m_n(\gamma)^{-\frac{1}{\gamma}}$ converges to zero in probability as $n \to +\infty$. As a consequence we will find that the weak limit of $\tilde{W}_n$ is a scale mixture of γ–stable laws, where the scale mixing measure is the law of $\tilde{M}_\infty(\gamma)^{\frac{1}{\gamma}}$. From the asymptotic results on $\tilde{W}_n$ we will easily deduce the weak convergence of the random vector (30) for $t \to +\infty$.

4.1. **The martingale of weights.** Let γ be a given positive real number such that $\mathbb{E}\Big[\sum_{i=1}^N A_i^\gamma\Big] < +\infty$. For every integer number $n \geq 1$ set

$$M_n(\gamma) := \sum_{v \in \mathcal{L}(T_n)} \varpi(v)^\gamma = \sum_{j=1}^{f_n} \beta_{j,n}^{\gamma}. \tag{33}$$

Note that

$$\tilde{M}_n(\gamma) = \frac{M_n(\gamma)}{m_n(\gamma)},$$

and, if $\mathcal{S}(\gamma) = 0$, then $\tilde{M}_n(\gamma) = M_n(\gamma)$.

The following proposition generalizes Lemma 2 in [2].

Proposition 4.1. *For every $\gamma > 0$ such that $\mathbb{E}[\sum_{i=1}^N A_i^\gamma] < +\infty$, one has*

$$\mathbb{E}[M_n(\gamma)] = m_n(\gamma) = \frac{\Big(\frac{\mathcal{S}(\gamma)+1}{N-1}\Big)_n}{\Big(\frac{1}{N-1}\Big)_n}$$

and, as $n \to +\infty$,

$$m_n(\gamma) = n^{\frac{\mathcal{S}(\gamma)}{N-1}} \frac{\Gamma(\frac{1}{N-1})}{\Gamma(\frac{\mathcal{S}(\gamma)+1}{N-1})} \Big(1 + O\Big(\frac{1}{n}\Big)\Big). \tag{34}$$

Moreover, $\tilde{M}_n(\gamma)$ is a positive martingale with respect to the filtration

$$\mathcal{G}_n = \sigma((A(v))_{v \in T_n}, T_1, \dots, T_n)$$

and $\mathbb{E}[\tilde{M}_n(\gamma)] = 1$*. Hence,* $\tilde{M}_n(\gamma)$ *converges almost surely to a random variable* $\tilde{M}_\infty(\gamma)$ *with* $\mathbb{E}[\tilde{M}_\infty(\gamma)] \leq 1$*.*

For every $\gamma > 0$, set

$$\beta_{(n)}^{(\gamma)} := \max_{v \in \mathcal{L}(T_n)} \frac{\varpi(v)}{m_n(\gamma)^{\frac{1}{\gamma}}} = \max_{j=1,\ldots,f_n} \frac{\beta_{j,n}}{m_n(\gamma)^{\frac{1}{\gamma}}}$$

and recall that $\mu(\gamma) = \mathcal{S}(\gamma)/\gamma$.

Proposition 4.2. *If for some* $\delta > 0$ *and* $\gamma > 0$ *one has* $\mu(\delta) < \mu(\gamma) < +\infty$*, then* $\beta_{(n)}^{(\gamma)}$ *converges in probability to* 0*. Moreover, if in addition* $\delta < \gamma$ *one has that* $\tilde{M}_n(\gamma)$ *converges almost surely to* 0*, that is* $\tilde{M}_\infty(\gamma) = 0$*. While, if* $\gamma < \delta$*,* $\tilde{M}_n(\gamma)$ *converges in* L^1 *to* $\tilde{M}_\infty(\gamma)$ *and hence* $\mathbb{E}[\tilde{M}_\infty(\gamma)] = 1$*.*

Proposition 4.3. *Assume that* $\mathbb{E}[\sum_{i=1}^N A_i^\gamma] < +\infty$*. Let* $\tilde{M}_\infty(\gamma)$ *be the same random variable defined in* Proposition 4.1 *and denote its characteristic function by* $\psi_{\infty,\gamma}$*. Then* $\psi_{\infty,\gamma}$ *satisfies the following integral equation*

$$\psi_{\infty,\gamma}(\xi) = \mathbb{E}\Big[\prod_{i=1}^N \psi_{\infty,\gamma}(A_i^\gamma U_i^{\frac{\mathcal{S}(\gamma)}{(N-1)}} \xi)\Big] \qquad (\xi \in \mathbb{R}) \tag{35}$$

where $U = (U_1, \ldots, U_N)$ *has Dirichlet distribution of parameters* $(1/(N-1), \ldots, 1/(N-1))$ *and* $(A_1, \ldots, A_N)$ *and* U *are stochastically independent.*

Note that (35) is equivalent to

$$M \stackrel{\mathcal{L}}{=} \sum_{i=1}^N A_i^\gamma U_i^{\frac{\mathcal{S}(\gamma)}{(N-1)}} M_i \tag{36}$$

where $(M, M_1, \ldots, M_n)$ are stochastically independent random variables with the same law of $\tilde{M}_\infty(\gamma)$, and $(M, M_1, \ldots, M_n)$, U and $(A_1, \ldots, A_N)$ are stochastically independent.

4.2. **Convergence of $\tilde{W}_n$ and of $(N_t(\gamma), \tilde{W}_{\nu_t})$.** Now we study the limiting distribution of $\tilde{W}_n$ defined by (31).

Proposition 4.4. *Let* (8) *be in force. Let* γ *belong to* $(0, 2]$ *and assume that there exists* $\delta > 0$ *such that* $\mu(\delta) < \mu(\gamma) < +\infty$*. Assume that condition* $(\mathbf{H}_\gamma)$ *holds true, then*

$$\lim_{n \to +\infty} \mathbb{E}[e^{i\xi \tilde{W}_n}] = \mathbb{E}[\hat{g}_\gamma(\tilde{M}_\infty(\gamma)^{\frac{1}{\gamma}} \xi)] \tag{37}$$

for every $\xi \in \mathbb{R}$*, where* $\tilde{M}_\infty(\gamma)$ *is the same random variable defined in* Proposition 4.1 *and* $\hat{g}_\gamma$ *is defined in* (11)*.*

At this stage, recall that $N_t(\gamma) = e^{-\mu(\gamma)t} m_{\nu_t}(\gamma)^{1/\gamma}$.

Proposition 4.5. *Under the assumptions of* Proposition 4.4

$$\lim_{t\to+\infty}\mathbb{E}[e^{i\xi_1 N_t(\gamma)+i\xi_2\tilde W_{\nu_t}}]=\mathbb{E}[e^{i\xi_1 c_\gamma Z^{\frac{\mu(\gamma)}{(N-1)}}}]\,\mathbb{E}[\hat g_\gamma(\tilde M_\infty(\gamma)^{\frac{1}{\gamma}}\xi_2)] \qquad (\xi_1,\xi_2)\in\mathbb{R}^2,$$

where Z has Gamma distribution with shape parameter $1/(N-1)$ and scale parameter 1,

$$c_\gamma := \left(\frac{\Gamma(\frac{1}{N-1})}{\Gamma(\frac{\mathcal{S}(\gamma)+1}{N-1})}\right)^{\frac{1}{\gamma}}, \tag{38}$$

$\tilde M_\infty(\gamma)$ is the same random variable defined in Proposition 4.1*, $\tilde M_\infty(\gamma)$ and Z are stochastically independent and $\hat g_\gamma$ is defined in* (11)*. As a consequence,*

$$\lim_{t\to+\infty}\phi(t,e^{-\mu(\gamma)t}\xi)=\mathbb{E}[\hat g_\gamma(c_\gamma Z^{\frac{\mu(\gamma)}{(N-1)}}\tilde M_\infty(\gamma)^{\frac{1}{\gamma}}\xi)] \qquad (\xi\in\mathbb{R}). \tag{39}$$

The result in equation (39) is the core of Theorems 2.2-2.3 presented in Section 2.1. The further properties of the limiting distribution are proved in Section 5.3.

5. Proofs

5.1. **Proofs of Section 3.**

Proof of Proposition 3.1. Let us first prove (26). Recall that $\mathcal{J}_1=\{(0,\dots,0)\}$ and for $\underline{i}=(0,\dots,0)$

$$P\{T_1\in C^1_{\underline{i}}\}=P\{|T_1^{(1)}|=0,\dots,|T_1^{(N)}|=0\}=1=p_1(\underline{i}).$$

For every $k\geq 1$ and $j=1,\dots,N$:

$$P\left\{|T^{(l)}_{k+1}|=|T^{(l)}_k| \quad l\neq j, \quad |T^{(j)}_{k+1}|=|T^{(j)}_k|+1 \;\middle|\; |T^{(l)}_k| \quad l=1,,\dots,N\right\}=\frac{f_{|T^{(j)}_k|}}{f_k}.$$

This means that the problem of evaluating probability (26) can be reduced to a Polya urn scheme, where one starts with N balls of N different colours and at each step a ball is randomly drawn from the urn and replaced with N balls of the same colour. Hence, for every $k\geq 2$ and $\underline{i}=(i_1,\dots i_N)\in\mathcal{J}_k$

$$P\left\{|T^{(l)}_k|=i_l, \quad l=1,N\right\}=\frac{(k-1)!}{\prod_{l=1}^N i_l!}\frac{\prod_{l=1}^N\prod_{m=0}^{i_l-1}f_m}{\prod_{r=0}^{k-1}f_r}=p_k(\underline{i})$$

which is (26).

Let us prove (25) by induction. For $k=1$ equality (25) is trivially true. Let us suppose (25) holds for k. Let $t\in\mathbb{T}_{k+1,N}$ and $\underline{i}=(i_1,\dots i_N)=(|t^{(1)}|,\dots,|t^{(N)}|)\in\mathcal{J}_{k+1}$, then

$$\begin{aligned}
&P\{T^{(1)}_{k+1}=t^{(1)},\dots,T^{(N)}_{k+1}=t^{(N)},T_{k+1}\in C^{k+1}_{\underline{i}}\}\\
&=P\{T^{(1)}_{k+1}=t^{(1)},\dots,T^{(N)}_{k+1}=t^{(N)}\}\\
&=\sum_{\substack{j=1,\dots,N\\ j:i_j\geq 1}}\sum_{t^*_j\in A_{j,t}}P\{T^{(1)}_{k+1}=t^{(1)},\dots,T^{(N)}_{k+1}=t^{(N)}|T^{(l)}_k=t^{(l)}\ \ l\neq j,\ \ T^{(j)}_k=t^*_j\}P\{T^{(l)}_k=t^{(l)}\ \ l\neq j,\ \ T^{(j)}_k=t^*_j\}
\end{aligned}$$

where $A_{j,t} = \{t_j^* \in \mathbb{T}_{i_j-1,N} \ : \ \exists\, v \in \mathcal{L}(t_j^*) \text{ such that } t_j^* \cup \{(v,1),\dots(v,N)\} = t^{(j)}\}$. By construction of a random N-ary tree, if $i_j \geq 1$ and $t_j^* \in A_{j,t}$,

$$P\{T_{k+1}^{(1)} = t^{(1)}, \dots, T_{k+1}^{(N)} = t^{(N)} | T_k^{(l)} = t^{(l)} \ \ l \neq j, \ \ T_k^{(j)} = t_j^*\} = \frac{1}{f_k}, \tag{40}$$

and

$$P\{T_{i_j} = t^{(j)}\} = \frac{1}{f_{i_j-1}} \sum_{t_j^* \in A_{j,t}} P\{T_{i_j-1} = t_j^*\}. \tag{41}$$

Furthermore, in view of the induction hypotheses and (26), one gets

$$P\{T_k^{(l)} = t^{(l)} \ \ l \neq j, \ \ T_k^{(j)} = t_j^*\} = \prod_{l=1}^N P\{T_{i_l} = t^{(l)}\} \frac{P\{T_{i_j-1} = t_j^*\}}{P\{T_{i_j} = t^{(j)}\}} p_k(i_1, \dots, i_j - 1, \dots, i_N). \tag{42}$$

Hence, from (40), (41) and (42), one obtains

$$\begin{aligned}
&P\{T_{k+1}^{(1)} = t^{(1)}, \dots, T_{k+1}^{(N)} = t^{(N)}\} \\
&= \prod_{l=1}^N P\{T_{i_l} = t^{(l)}\} \sum_{\substack{j=1,\dots,N \\ j: i_j \geq 1}} \frac{1}{f_k} p_k(i_1, \dots, i_j - 1, \dots, i_N) \frac{1}{P\{T_{i_j} = t^{(j)}\}} \sum_{t_j^* \in A_{j,t}} P\{T_{i_j-1} = t_j^*\} \\
&= \prod_{l=1}^N P\{T_{i_l} = t^{(l)}\} \sum_{\substack{j=1,\dots,N \\ j: i_j \geq 1}} \frac{f_{i_j-1}}{f_k} p_k(i_1, \dots, i_j - 1, \dots, i_N) \\
&= \prod_{l=1}^N P\{T_{i_l} = t^{(l)}\} p_{k+1}(i_1, \dots, i_N),
\end{aligned}$$

where the last equality is obtained by direct replacement of the espression of $p_k(i_1, \dots, i_j - 1, \dots, i_N)$. Note that, using the Polya urn interpretation, $|T_k^{(l)}|$ represents the numbers of balls of color l drawn in the first $k-1$ steps. Hence, using the results in [4], the almost sure convergence of $(|T_k^{(l)}|/(k-1) : l = 1, \dots, N)$ follows by the strong law of large numbers for exchangeable sequences. □

Proof of Proposition 3.2. We need only to prove that $q_n(\xi) = \mathbb{E}[e^{i\xi W_n}]$, for every $n \geq 0$. This is clearly true for $n = 0$. For $n \geq 1$, write

$$W_n = \sum_{j=1}^N A_j(\emptyset) \Big[\sum_{v \in \mathcal{L}(T_n^{(j)})} \prod_{i=0}^{|v|-1} A_{v_{i+1}}^{(j)}(v|i) X_v^{(j)} \Big]$$

where $A^{(j)}(v) = A((j,v))$, $X_v^{(j)} = X_{(j,v)}$ and, by convention, if $\mathcal{L}(T_n) = \emptyset$ the term between square brakets is equal to X_j. Since $(A^{(j)}(v), X_v^{(j)})_{v \in \mathbb{U}}$, $j = 1, \dots, N$, are independent, with the same distribution of $(A(v), X_v)_{v \in \mathbb{U}}$, using (25) and the induction hypothesis one proves that

$$\mathbb{E}\left[e^{i\xi W_n} \Big| A(\emptyset), |T_n^{(1)}|, \dots, |T_n^{(N)}| \right] = \prod_{j=1}^N q_{|T_n^{(j)}|}(\xi A_j(\emptyset)). \tag{43}$$

At this stage the conclusion follows easily by using (26); indeed:

$$\mathbb{E}[e^{i\xi W_n}] = \mathbb{E}\Big[\prod_{j=1}^{N} q_{|T_n^{(j)}|}(\xi A_j(\emptyset))\Big] = \sum_{\underline{i}\in\mathcal{J}_n} \mathbb{E}\Big[\prod_{j=1}^{N} q_{i_j}(\xi A_j)\Big] p_n(\underline{i}) = q_n(\xi).$$

□

5.2. **Proofs of Section 4.**

Proof of Proposition 4.1. Clearly $\varpi(v)\mathbb{I}\{v \in \mathcal{L}(T_n)\}$ is $\mathcal{G}_n$–measurable. We first prove that

$$\mathbb{E}[M_{n+1}(\gamma)|\mathcal{G}_n] = M_n(\gamma)(1+\mathcal{S}(\gamma)/f_n). \tag{44}$$

Given a sequence $(T_n)_{n\geq 1}$ of random N-ary recursive trees, one can define a sequence $(V_n)_{n\geq 1}$ of $\mathbb{U}$-valued random variables such that

$$T_{n+1} = T_n \cup \{(V_n,1),\dots,(V_n,N)\}$$

for every $n \geq 0$, where $V_0 = \emptyset$ and $V_n \in \mathcal{L}(T_n)$. The random variable V_n corresponds to the random vertex chosen to generate T_{n+1} from T_n. Hence, by construction, $P(V_n = v|T_1,\dots,T_n) = \mathbb{I}\{v \in \mathcal{L}(T_n)\}1/f_n$ for every $n \geq 1$ and $P(V_n = v|\mathcal{G}_n) = 1/f_n\mathbb{I}\{v \in \mathcal{L}(T_n)\}$. At this stage one can write

$$\begin{aligned}
&\mathbb{E}[M_{n+1}(\gamma)|\mathcal{G}_n] = M_n(\gamma)\\
&+ \mathbb{E}\Big[\sum_{v\in\mathcal{L}(T_n)} \varpi(v)^\gamma \Big(A_1(v)^\gamma + \dots + A_N(v)^\gamma - 1\Big)\mathbb{I}\{V_n = v\}\Big|\mathcal{G}_n\Big]\\
&= M_n(\gamma) + \mathcal{S}(\gamma) \sum_{v\in\mathcal{L}(T_n)} \varpi(v)^\gamma \mathbb{E}[\mathbb{I}\{V_n = v\}|\mathcal{G}_n] = M_n(\gamma)(1+\mathcal{S}(\gamma)/f_n).
\end{aligned}$$

Taking the expectation of both sides gives $\mathbb{E}[M_{n+1}(\gamma)] = \mathbb{E}[M_n(\gamma)](1+\mathcal{S}(\gamma)/f_n)$. Since $\mathbb{E}[M_1(\gamma)] = \mathcal{S}(\gamma)+1$ and $f_0 = 1$ it follows immediately that

$$\mathbb{E}[M_n(\gamma)] = \prod_{i=0}^{n-1}(1+\mathcal{S}(\gamma)/f_i) = m_n(\gamma). \tag{45}$$

See (32). Since $f_i = (N-1)i+1$, by simple algebra one gets that

$$m_n(\gamma) = \frac{\Gamma\Big(\frac{\mathcal{S}(\gamma)+1}{N-1}+n\Big)\Gamma\Big(\frac{1}{N-1}\Big)}{\Gamma\Big(\frac{\mathcal{S}(\gamma)+1}{N-1}\Big)\Gamma\Big(\frac{1}{N-1}+n\Big)} = \frac{\Big(\frac{\mathcal{S}(\gamma)+1}{N-1}\Big)_n}{\Big(\frac{1}{N-1}\Big)_n}.$$

At this stage, recall that given two positive real numbers x and y

$$\frac{\Gamma(x+n)}{\Gamma(y+n)} = n^{x-y}\Big(1+O\Big(\frac{1}{n}\Big)\Big)$$

as $n \to +\infty$, which proves (34). Finally, (44)-(45) yield that $\tilde{M}_n(\gamma)$ is a $(\mathcal{G}_n)_n$–martingale since $M_n(\gamma) \geq 0$ and $\mathbb{E}[M_n(\gamma)] < +\infty$ for every $n \geq 1$. The last part of the theorem follows by classical martingale theory. □

Proof of Proposition 4.2. Observe that

$$[\beta_{(n)}^{(\gamma)}]^\delta \leq \sum_{v\in\mathcal{L}(T_n)} \frac{\varpi(v)^\delta}{m_n(\gamma)^{\frac{\delta}{\gamma}}},$$

hence for every $\epsilon > 0$, by Markov's inequality and (34), one gets

$$\begin{aligned} P\{\beta_{(n)}^{(\gamma)} > \epsilon\} &\leq P\Big\{ \sum_{v\in\mathcal{L}(T_n)} \frac{\varpi(v)^\delta}{m_n(\gamma)^{\frac{\delta}{\gamma}}} \geq \epsilon^\delta \Big\} \leq \frac{1}{\epsilon^\delta m_n(\gamma)^{\delta/\gamma}} \mathbb{E}[M_n(\delta)] = \frac{m_n(\delta)}{\epsilon^\delta m_n(\gamma)^{\delta/\gamma}} \\ &\leq \frac{C_{\delta,\gamma}}{\epsilon^\delta} n^{\frac{\delta}{N-1}(\mathcal{S}(\delta)/\delta - \mathcal{S}(\gamma)/\gamma)} = \frac{C_{\delta,\gamma}}{\epsilon^\delta} n^{\frac{\delta}{N-1}(\mu(\delta)-\mu(\gamma))}. \end{aligned}$$

This proves the first statement. When $\delta < \gamma$, one has $\delta/\gamma < 1$ and hence, using Minkowski inequality and (34), one gets

$$\begin{aligned} \mathbb{E}[\tilde{M}_n(\gamma)^{\delta/\gamma}] &\leq \frac{1}{m_n(\gamma)^{\delta/\gamma}} \mathbb{E}\Big[\sum_{v\in\mathcal{L}(T_n)} \varpi(v)^\delta \Big] \\ &\leq \frac{m_n(\delta)}{m_n(\gamma)^{\delta/\gamma}} \leq C_{\delta,\gamma} n^{\frac{\delta}{N-1}(\mu(\delta)-\mu(\gamma))}, \end{aligned}$$

which proves the second statement. Assume now that $\delta > \gamma$. In order to prove the last part of the statement let us show that $\tilde{M}_n(\gamma)$ is uniformly integrable. To this end, observe that

$$\tilde{M}_{n+1}(\gamma) = \sum_{v\in\mathcal{L}(T_n)} \frac{\varpi(v)^\gamma}{m_n(\gamma)} \frac{\Big[1 + \Big(\sum_{k=1}^N A_k(v)^\gamma - 1\Big)\mathbb{I}\{V_n = v\}\Big]}{1 + \frac{\mathcal{S}(\gamma)}{f_n}}$$

and hence

$$\tilde{M}_{n+1}(\gamma) - \tilde{M}_n(\gamma) = -\frac{\tilde{M}_n(\gamma)\mathcal{S}(\gamma)}{f_n(1+\mathcal{S}(\gamma)/f_n)} + \frac{1}{m_{n+1}(\gamma)} \sum_{v\in\mathcal{L}(T_n)} \varpi(v)^\gamma \Big(\sum_{k=1}^N A_k(v)^\gamma - 1 \Big)\mathbb{I}\{V_n = v\}.$$

At this stage write

$$\begin{aligned} |\tilde{M}_{n+1}(\gamma) - \tilde{M}_n(\gamma)|^{\frac{\delta}{\gamma}} &\leq 2^{\frac{\delta}{\gamma}-1} \frac{|\tilde{M}_n(\gamma)|^{\frac{\delta}{\gamma}} |\mathcal{S}(\gamma)|^{\frac{\delta}{\gamma}}}{f_n^{\frac{\delta}{\gamma}} |1+\mathcal{S}(\gamma)/f_n|^{\frac{\delta}{\gamma}}} \\ &\quad + \frac{2^{\frac{\delta}{\gamma}-1}}{m_{n+1}(\gamma)^{\frac{\delta}{\gamma}}} \sum_{v\in\mathcal{L}(T_n)} \varpi(v)^\delta \Big| \sum_{k=1}^N A_k(v)^\gamma - 1 \Big|^{\frac{\delta}{\gamma}} \mathbb{I}\{V_n = v\} \\ &\leq \frac{2^{\frac{\delta}{\gamma}-1} |\mathcal{S}(\gamma)|^{\frac{\delta}{\gamma}}}{|1+\mathcal{S}(\gamma)/f_n|^{\frac{\delta}{\gamma}}} \frac{1}{f_n^{\frac{\delta}{\gamma}}} f_n^{\frac{\delta}{\gamma}-1} \sum_{v\in\mathcal{L}(T_n)} \frac{\varpi(v)^\delta}{m_n(\gamma)^{\frac{\delta}{\gamma}}} \\ &\quad + \frac{2^{\frac{\delta}{\gamma}-1}}{m_{n+1}(\gamma)^{\frac{\delta}{\gamma}}} \sum_{v\in\mathcal{L}(T_n)} \varpi(v)^\delta \Big| \sum_{k=1}^N A_k(v)^\gamma - 1 \Big|^{\frac{\delta}{\gamma}} \mathbb{I}\{V_n = v\}. \end{aligned}$$

Taking the expectation one gets

$$
\begin{aligned}
\mathbb{E}|\tilde{M}_{n+1}(\gamma)-\tilde{M}_n(\gamma)|^{\frac{\delta}{\gamma}} &\leq \frac{2^{\frac{\delta}{\gamma}-1}|\mathcal{S}(\gamma)|^{\frac{\delta}{\gamma}}}{|1+\mathcal{S}(\gamma)/f_n|^{\frac{\delta}{\gamma}}}\frac{1}{f_n}\mathbb{E}\Big[\sum_{v\in\mathcal{L}(T_n)}\frac{\varpi(v)^\delta}{m_n(\gamma)^{\frac{\delta}{\gamma}}}\Big] \\
&+\frac{2^{\frac{\delta}{\gamma}-1}}{m_{n+1}(\gamma)^{\frac{\delta}{\gamma}}}\frac{1}{f_n}\mathbb{E}\Big[\sum_{v\in\mathcal{L}(T_n)}\varpi(v)^\delta\Big]\mathbb{E}\Big|\sum_{k=1}^N A_k^\gamma-1\Big|^{\frac{\delta}{\gamma}} \\
&\leq C_1\frac{1}{f_n}\Big[\frac{m_n(\delta)}{m_n(\gamma)^{\delta/\gamma}}+\frac{m_n(\delta)}{m_{n+1}(\gamma)^{\delta/\gamma}}\Big] \\
&\qquad\text{by (34)} \\
&\leq C_2\frac{1}{f_n}n^{\frac{\delta(\mu(\delta)-\mu(\gamma))}{N-1}}\leq C_3 n^{\frac{\delta(\mu(\delta)-\mu(\gamma))}{N-1}-1}.
\end{aligned}
$$

Since $\mu(\delta)<\mu(\gamma)$, it follows that

$$
\sum_{i\geq 1}\mathbb{E}[|\tilde{M}_{i+1}(\gamma)-\tilde{M}_i(\gamma)|^{\frac{\delta}{\gamma}}]<+\infty. \tag{46}
$$

By the convexity of $\mathcal{S}(s)$ it is easy to see that $\mu(s)<\mu(\gamma)$ if $\gamma<s<\delta$. Hence, without loss of generality, one can suppose that $\gamma<\delta\leq 2\gamma$. Since $(\tilde{M}_n)_{n\geq 1}$ is a martingale (cf. Proposition 4.1) and $1<\delta/\gamma\leq 2$, the Topchii-Vatutin inequality (see e.g. [1]) gives

$$
\mathbb{E}|\tilde{M}_n(\gamma)|^{\frac{\delta}{\gamma}}\leq\mathbb{E}|\tilde{M}_1(\gamma)|^{\frac{\delta}{\gamma}}+2\sum_{i=2}^n\mathbb{E}|\tilde{M}_i(\gamma)-\tilde{M}_{i-1}(\gamma)|^{\frac{\delta}{\gamma}}
$$

Combining this last inequality with (46) one obtains

$$
\sup_{n\geq 1}\mathbb{E}|\tilde{M}_n(\gamma)|^{\delta/\gamma}<+\infty.
$$

Hence $(\tilde{M}_n(\gamma))_n$ is uniformly integrable and then converges in L^1 to $\tilde{M}_\infty(\gamma)$ with $\mathbb{E}[\tilde{M}_\infty(\gamma)]=\lim_n\mathbb{E}[\tilde{M}_n(\gamma)]=1$. □

Proof of Proposition 4.3. Let $\psi_n(\xi)=\mathbb{E}[e^{i\xi\tilde{M}_n(\gamma)}]$. Arguing as in the proof of Proposition 3.2 and using the same notation, we get

$$
\tilde{M}_n(\gamma)=\sum_{j=1}^N A_j(\emptyset)^\gamma\frac{m_{|T_n^{(j)}|}(\gamma)}{m_n(\gamma)}\sum_{v\in\mathcal{L}(T_n^{(j)})}\frac{\prod_{i=0}^{|v|-1}\big(A_{v_{i+1}}^{(j)}(v|i)\big)^\gamma}{m_{|T_n^{(j)}|}(\gamma)},
$$

and then

$$
\psi_n(\xi)=\mathbb{E}\Big[\prod_{j=1}^N\psi_{|T_n^{(j)}|}\Big(\xi A_j^\gamma\Delta_n^{(j)}\Big)\Big]
$$

where

$$
\Delta_n^{(j)}=\frac{m_{|T_n^{(j)}|}(\gamma)}{m_n(\gamma)}=\Big(\frac{|T_n^{(j)}|}{n}\Big)^{\frac{\mathcal{S}(\gamma)}{N-1}}\Big(\frac{1+O\big(\frac{1}{|T_n^{(j)}|}\big)}{1+O\big(\frac{1}{n}\big)}\Big).
$$

Now note that, for a suitable constant C, $\Delta_n^{(j)}\leq C$ for every j almost surely and, by Proposition 3.1, $(\Delta_n^{(1)},\dots,\Delta_n^{(N)})$ converges almost surely to $(U_1^{\frac{\mathcal{S}(\gamma)}{(N-1)}},\dots,U_N^{\frac{\mathcal{S}(\gamma)}{(N-1)}})$ where $(U_1,\dots,U_N)$ has

Dirichlet distribution of parameters $(1/(N-1), \dots, 1/(N-1))$. At this stage write

$$\psi_n(\xi) = \mathbb{E}\Big[\prod_{j=1}^{N} \psi_{\infty,\gamma}\Big(\xi A_j^\gamma \Delta_n^{(j)}\Big)\Big] + R_n$$

with

$$R_n = \mathbb{E}\Big[\prod_{j=1}^{N} \psi_{|T_n^{(j)}|}\Big(\xi A_j^\gamma \Delta_n^{(j)}\Big) - \prod_{j=1}^{N} \psi_{\infty,\gamma}\Big(\xi A_j^\gamma \Delta_n^{(j)}\Big)\Big].$$

By dominated convergence one gets

$$\lim_{n\to+\infty} \mathbb{E}\Big[\prod_{j=1}^{N} \psi_{\infty,\gamma}\Big(\xi A_j^\gamma \Delta_n^{(j)}\Big)\Big] = \mathbb{E}\Big[\prod_{j=1}^{N} \psi_{\infty,\gamma}\Big(\xi A_j^\gamma U_j^{\frac{\mathcal{S}(\gamma)}{(N-1)}}\Big)\Big].$$

It remains to show that R_n converges to zero. Recall that, given $2N$ complex numbers $a_1, \dots, a_N$,b_1,$\dots$, b_N with $|a_i|, |b_i| \le 1$, then

$$|\prod_{i=1}^{N} a_i - \prod_{i=1}^{N} b_i| \le \sum_{i=1}^{N} |a_i - b_i|.$$

Hence

$$\begin{aligned}
|R_n| &\le \sum_{j=1}^{N} \mathbb{E}\Big|\psi_{|T_n^{(j)}|}\Big(\xi A_j^\gamma \Delta_n^{(j)}\Big) - \psi_{\infty,\gamma}\Big(\xi A_j^\gamma \Delta_n^{(j)}\Big)\Big| \\
&\le \sum_{j=1}^{N} \mathbb{E}\Big[\sup_{x:|x|\le|\xi A_j^\gamma C|} \Big|\psi_{|T_n^{(j)}|}(x) - \psi_{\infty,\gamma}(x)\Big|\Big].
\end{aligned}$$

Since point-wise convergence of characteristic functions yields the same convergence on every compact set and ψ_n converges to $\psi_{\infty,\gamma}$, one has that $\sup_{x:|x|\le C} |\psi_n(x) - \psi_{\infty,\gamma}(x)|$ converges to zero when n goes to $+\infty$ for every $C < +\infty$. By Proposition 3.1 $|T_n^{(j)}|$ converges almost surely to $+\infty$, hence dominated convergence yields that $\sup_{x:|x|\le|\xi A_j^\gamma C|} \big|\psi_{|T_n^{(j)}|}(x) - \psi_{\infty,\gamma}(x)\big|$ converges almost surely to zero as n goes to $+\infty$ and then, by dominated convergence, R_n converges to zero. □

In order to prove Proposition 4.4 let us consider the following central limit result. Assume that $(a_{jn})_{jn}$ is an array of positive weights and let f_n be a diverging sequence of integer numbers. Given any array of identically distributed and row-wise independent random variables $(X_{jn})_{n\ge1, j=1,\dots,f_n}$ with probability distribution function F_0, set

$$S_n = \sum_{j=1}^{f_n} a_{jn} X_{j,n}.$$

Moreover assume that, for some γ in $(0, 2]$,

$$\lim_{n\to+\infty} \sum_{j=1}^{f_n} a_{jn}^\gamma = a_\infty, \qquad \text{and} \qquad \lim_{n\to+\infty} \max_{j=1,\dots,f_n} a_{jn} = 0. \tag{47}$$

It is not hard to prove the following

Lemma 5.1. *Let* (47) *and* $(\mathbf{H}_\gamma)$ *be in force for* γ *in* $(0,2]$. *Then*

$$\lim_{n\to+\infty} \mathbb{E}[e^{i\xi S_n}] = \hat{g}_\gamma(\xi a_\infty^{\frac{1}{\gamma}}) \tag{48}$$

for every $\xi \in \mathbb{R}$, $\hat{g}_\gamma$ *being defined in* (11).

Proof. The proof can be obtained following the same line of the proofs of Lemmata 4,5,6 in [2] as a consequence of the central limit theorem for triangular array. See, e.g., [11]. □

Proof of Proposition 4.4. Denote by $\mathcal{B}$ the σ-algebra generated by $\{T_n, \beta_{j,n} : n \geq 1, j = 1,\dots,f_n\}$. The proof is essentially an application of Lemma 5.1 to the conditional law of $\tilde{W}_n$ given $\mathcal{B}$. By Propositions 4.1-4.2, every divergent sequence (n') of integer numbers contains a divergent subsequence $(n'') \subset (n')$ for which $\tilde{M}_{n''}(\gamma)$ converges almost surely to $\tilde{M}_\infty(\gamma)$ and $\beta_{(n'')}^{(\gamma)}$ converges almost surely to zero. Hence by Lemma 5.1 we obtain $\lim_{n''\to+\infty} \mathbb{E}[e^{i\xi \tilde{W}_{n''}}|\mathcal{B}] = \hat{g}_\gamma(\xi \tilde{M}_\infty^{\frac{1}{\gamma}}(\gamma))$ almost surely. Dominated convergence theorem yields $\lim_{n''\to+\infty} \mathbb{E}[e^{i\xi\tilde{W}_{n''}}] = \mathbb{E}[\hat{g}_\gamma(\xi\tilde{M}_\infty^{\frac{1}{\gamma}}(\gamma))]$. Since the limiting function is independent of the arbitrarily chosen sequence (n'), a classical argument shows that the last limit is true with $n \to +\infty$ in place of $n'' \to +\infty$. □

Proof of Proposition 4.5. Let us first prove that when t goes to $+\infty$, $\nu_t e^{-t(N-1)}$ converges in distribution to a random variable Z with Gamma distribution of parameters $(1/(N-1), 1)$. Since ν_t is a negative-binomial random variable of parameters $(1/(N-1), \exp\{-(N-1)t\})$, for every integer k

$$P\{\nu_t \leq k\} = \frac{\Gamma(k+1+\frac{1}{N-1})}{\Gamma(k+1)\Gamma(1/(N-1))} \int_0^{e^{-(N-1)t}} x^{\frac{1}{N-1}-1}(1-x)^k dx.$$

See formula (5.31) in [14]. Hence, for every $y > 0$, after setting $k_t = \lfloor ye^{(N-1)t} \rfloor$ (where $\lfloor x \rfloor$ is the integer part of x), one can write

$$\begin{aligned}
P\{\nu_t e^{-(N-1)t} \leq y\} &= P\{\nu_t \leq k_t\} \\
&= \frac{\Gamma(k_t+1+\frac{1}{N-1})}{\Gamma(k_t+1)\Gamma(1/(N-1))} \int_0^{e^{-(N-1)t}} x^{\frac{1}{N-1}-1}(1-x)^{k_t} dx \\
&= \frac{\Gamma(k_t+1+\frac{1}{N-1})}{\Gamma(k_t+1)\Gamma(1/(N-1))} \frac{1}{[ye^{(N-1)t}]^{\frac{1}{N-1}}} \int_0^y u^{\frac{1}{N-1}-1}\Big(1-\frac{u}{ye^{(N-1)t}}\Big)^{k_t} du \\
&= \frac{1+O(1/k_t)}{\Gamma(1/N-1)} \Big[\frac{k_t}{ye^{(N-1)t}}\Big]^{\frac{1}{N-1}} \int_0^y u^{\frac{1}{N-1}-1}\Big(1-\frac{u}{ye^{(N-1)t}}\Big)^{ye^{(N-1)t}\frac{k_t}{ye^{(N-1)t}}} du.
\end{aligned}$$

Since $k_t/ye^{(N-1)t} \to 1$, by dominated convergence one gets

$$\lim_{t\to+\infty} P\{\nu_t e^{-(N-1)t} \leq y\} = \frac{1}{\Gamma(\frac{1}{N-1})} \int_0^y u^{\frac{1}{N-1}-1} e^{-u} du.$$

At this stage, since ν_t converges in probability to $+\infty$, (34), Slustky theorem and the continuous mapping theorem yield that

$$\lim_{t\to+\infty} \mathbb{E}[e^{i\xi N_t(\gamma)}] = \mathbb{E}[e^{i\xi c_\gamma Z^{\frac{\mu(\gamma)}{(N-1)}}}]. \tag{49}$$

for every ξ in $\mathbb{R}$. Setting $u_n(\xi) := \mathbb{E}[e^{i\xi \tilde{W}_n}]$ by Proposition 4.4 we know that

$$\lim_{n\to+\infty} u_n(\xi) = \mathbb{E}[\hat{g}_\gamma(\xi \tilde{M}_\infty(\gamma)^{\frac{1}{\gamma}})] =: u_\infty(\xi) \tag{50}$$

for every ξ in $\mathbb{R}$. For every diverging sequence $(t_n)_n$ write

$$\mathbb{E}[e^{i\xi_1 N_{t_n}(\gamma)+i\xi_2 \tilde{W}_{\nu_{t_n}}}] = u_\infty(\xi_2)\mathbb{E}[e^{i\xi_1 N_{t_n}(\gamma)}] + R_n$$

where

$$R_n = \mathbb{E}[e^{i\xi_1 N_{t_n}(\gamma)}\big(e^{i\xi_2 \tilde{W}_{\nu_{t_n}}} - u_\infty(\xi_2)\big)].$$

It is easy to show that

$$\lim_{n\to+\infty} |R_n| \le \lim_{n\to+\infty} \mathbb{E}|u_{\nu_{t_n}}(\xi_2) - u_\infty(\xi_2)| = 0$$

by dominated convergence, since ν_{t_n} converges in probability to $+\infty$ and (50) holds. The result now follows from (49). The second part of the statement follows immediately by the continuous mapping theorem. $\square$

5.3. **Proofs of Section 2.** Under the hypotheses of Proposition 4.4, (39) yields that, when $\gamma \neq 1$ or when $\gamma = 1$ and condition (b) of $(\mathbf{H}_1)$ holds, $e^{-\mu(\gamma)t}V_t$ converges in distribution to a scale mixture of Stable laws. The scale mixing measure is the law of $c_\gamma Z^{\frac{\mu(\gamma)}{N-1}} \tilde{M}_\infty(\gamma)^{\frac{1}{\gamma}}$, with Z and $\tilde{M}_\infty(\gamma)$ stochastically independent. While if $\gamma = 1$ and condition (a) of $(\mathbf{H}_1)$ holds, then $e^{-\mu(\gamma)t}V_t$ converges in distribution to $m_0 c_1 Z^{\frac{\mu(1)}{N-1}} \tilde{M}_\infty(1)$. Again the mixing measure is the law of $c_1 Z^{\frac{\mu(1)}{N-1}} \tilde{M}_\infty(1)$. At this stage, in order to complete the proof of the main theorems of Section 2.1, it remains to study more in details the distribution of $c_\gamma Z^{\frac{\mu(\gamma)}{N-1}} \tilde{M}_\infty(\gamma)^{\frac{1}{\gamma}}$.

The more interesting case is $\mu(\delta) < \mu(\gamma)$ for $\delta > \gamma$. Proposition 4.3 shows that the law of $\tilde{M}_\infty(\gamma)$ satisfies the fixed point equation for distribution (36). We will show that the law of $c_\gamma^\gamma Z^{\frac{\mathcal{S}(\gamma)}{N-1}} \tilde{M}_\infty(\gamma)$ and the limit law of $e^{-\mu(\gamma)t}V_t$ satisfy a fixed point equation too. In view of known results on this kind of equations, we will be able to conclude the proof of Theorem 2.2 of Section 2.1.

In what follows denote by $Beta(a,b)$ ($Gamma(a,b)$, respectively), $a > 0$ and $b > 0$, the beta distribution of parameters a and b (the gamma distribution of shape parameter a and scale parameter b, respectively). We will need the following result.

Lemma 5.2. *Let $Z_1, \dots, Z_N, V$ be independent random variables where V has $Beta(1/(N-1), 1)$ distribution and Z_i has $Gamma(1/(N-1), 1)$ distribution for every i. Let Z and $U = (U_1, \dots, U_N)$ be stochastically independent, Z with $Gamma(1/(N-1), 1)$ distribution and U with Dirichlet distribution of parameters $(1/(N-1), \dots, 1/(N-1))$. Then*

$$(ZU_1, ZU_2, \dots, ZU_N) \stackrel{\mathcal{L}}{=} (VZ_1, VZ_2, \dots, VZ_N).$$

Proof. Set $S = \sum_{i=1}^N Z_i$. S is a $Gamma(N/(N-1),1)$ random variable and S and V are stochastically independent. It is easily seen that SV is a $Gamma(1/(N-1),1)$ random variable. Now define $\tilde U := (Z_1/S, \dots, Z_N/S)$ and $\tilde Z := SV$. It is well known that $\tilde U$ has a Dirichlet distribution of parameters $(1/(N-1), \dots, 1/(N-1)$ and S and $\tilde U$ are independent. See, e.g., Section 10.4 in [11]. Hence, it turns out that $\tilde U$ and $\tilde Z$ are stochastically independent. Clearly

$$(\tilde Z \tilde U_1, \dots, \tilde Z \tilde U_N) = (VZ_1, \dots, VZ_N)$$

which proves the claim. □

Proposition 5.3. *Let the assumptions of* Proposition 4.4 *be in force and let $v_{\infty,\gamma}$ be the characteristic function of $c_\gamma^\gamma Z^{\frac{\mathcal{S}(\gamma)}{N-1}} \tilde M_\infty(\gamma)$. Then $v_{\infty,\gamma}$ satisfies the integral equation* (13)*, that is*

$$v_{\infty,\gamma}(\xi) = \int_0^1 \mathbb{E}\left[\prod_{i=1}^N v_{\infty,\gamma}(\xi A_i^\gamma \tau^{\mathcal{S}(\gamma)})\right] d\tau. \tag{51}$$

Moreover, if $w_{\infty,\gamma}$ denotes the characteristic function of the limit in distribution of $e^{-\mu(\gamma)t}V_t$, then $w_{\infty,\gamma}$ satisfies equation (6) *for $\mu^* = \mu(\gamma)$, that is*

$$w_{\infty,\gamma}(\xi) = \int_0^1 \mathbb{E}\Big[\prod_{i=1}^N w_{\infty,\gamma}(A_i \tau^{\mu(\gamma)} \xi)\Big] d\tau = \int_0^1 \widehat{Q}[w_{\infty,\gamma}](\tau^{\mu(\gamma)}\xi) d\tau. \tag{52}$$

Proof. Recall that $\psi_{\infty,\gamma}$ denotes the characteristic function of $\tilde M_\infty(\gamma)$. Hence from the independence of Z and $\tilde M_\infty(\gamma)$ we have

$$v_{\infty,\gamma}(\xi) = \mathbb{E}[\psi_{\infty,\gamma}(\xi c_\gamma^\gamma Z^{\frac{\mathcal{S}(\gamma)}{N-1}})].$$

Since $\psi_{\infty,\gamma}$ satisfies equation (35) we can write

$$v_{\infty,\gamma}(\xi) = \mathbb{E}\Big[\prod_{i=1}^N \psi_{\infty,\gamma}(A_i^\gamma (U_i Z)^{\frac{\mathcal{S}(\gamma)}{N-1}} c_\gamma^\gamma \xi)\Big]$$

where $U = (U_1, \dots, U_N)$, $(A_1, \dots, A_N)$ and Z are independent, U has Dirichlet distribution of parameters $(1/(N-1), \dots, 1/(N-1))$ and Z has $Gamma(1/(N-1),1)$ distribution. By Lemma 5.2 if $(Z_1, \dots, Z_N, V)$ are independent random variables, Z_i with $Gamma(1/(N-1),1)$ distribution, V with $Beta(1/(N-1),1)$ distribution and $(Z_1, \dots, Z_N, V)$ and $(A_1, \dots, A_N)$ independent, then we can write

$$v_{\infty,\gamma}(\xi) = \mathbb{E}\Big[\prod_{i=1}^N \psi_{\infty,\gamma}(A_i^\gamma (VZ_i)^{\frac{\mathcal{S}(\gamma)}{N-1}} c_\gamma^\gamma \xi)\Big] = \mathbb{E}\Big[\prod_{i=1}^N v_{\infty,\gamma}(A_i^\gamma V^{\frac{\mathcal{S}(\gamma)}{N-1}} \xi)\Big].$$

Then (51) follows since $V^{1/(N-1)}$ has uniform distribution on $(0,1)$.

As for the second part, let $\hat g_\gamma$ be defined in (11). From Proposition 4.5 we know that

$$w_{\infty,\gamma}(\xi) = \mathbb{E}\Big[\hat g_\gamma(c_\gamma Z^{\frac{\mu(\gamma)}{N-1}} \tilde M_\infty(\gamma)^{\frac{1}{\gamma}})\Big].$$

Define $Y = c_\gamma^\gamma Z^{\frac{\mathcal{S}(\gamma)}{N-1}} \tilde{M}_\infty(\gamma)$, then (51) is equivalent to

$$Y \stackrel{\mathcal{L}}{=} \Theta^{\mathcal{S}(\gamma)} \sum_{i=1}^{N} A_i^\gamma Y_i \tag{53}$$

where $(Y, Y_1, \dots, Y_N)$ are i.i.d., Θ has uniform distribution on $(0,1)$ and $(Y, Y_1, \dots, Y_N)$, Θ and $(A_1, \dots, A_N)$ are independent. Then from (53) and from the analytic form of $\hat{g}_\gamma$ we get

$$\begin{aligned} w_{\infty,\gamma}(\xi) &= \mathbb{E}\Big[\hat{g}_\gamma(\xi Y^{\frac{1}{\gamma}})\Big] = \mathbb{E}\Big[\prod_{i=1}^{N} \hat{g}_\gamma(\xi \Theta^{\mu(\gamma)} A_i Y_i^{\frac{1}{\gamma}})\Big] \\ &= \mathbb{E}\Big[\prod_{i=1}^{N} w_{\infty,\gamma}(\xi \Theta^{\mu(\gamma)} A_i)\Big] \end{aligned}$$

.

and this completes the proof. $\square$

In order to prove Proposition 2.1 we need to recall few important results on fixed point equations for distributions. Assume that $B = (B_1, \dots, B_N)$ is a vector of non-negative random variables. Consider the following fixed point equation

$$\nu = T(\nu) \tag{54}$$

where, given any probability distribution ν, $T(\nu)$ is the law of $\sum_{i=1}^N B_i Y_i$, where B and $(Y_1, \dots, Y_N)$ are stochastically independent and $(Y_1, \dots, Y_N)$ are stochastically independent and identically distributed random variables with distribution ν. Clearly, (54) is equivalent to equation

$$\phi(\xi) = \mathbb{E}\left[\prod_{i=1}^{N} \phi(B_i \xi)\right] \tag{55}$$

for the corresponding Fourier-Stieltjes transforms. Equations (35),(13)-(51) and (52) have this form for a suitable B. In order to describe the fixed points of (54) we introduce the convex function $a : [0, \infty) \to [0, \infty]$ by

$$a(s) := \mathbb{E}\left[\sum_{j=1}^{N} B_j^s\right], \tag{56}$$

with the convention that $0^0 = 0$.

Proposition 5.4 ([9],[18],[19]). *Assume that condition* (8) *holds true with* B_i *in place of* A_i *and that* $a(1) = 1$.

(i) *If* $\sum_{j=1}^N B_j = 1$ *almost surely, then* $a(s) \geq 1$ *for every* $s < 1$ *and* $a(s) \leq 1$ *for every* $s > 1$. *Moreover, the unique solution* ζ *of equation* (54) *with* $\int_{\mathbb{R}^+} \zeta(dv) = \int_{\mathbb{R}^+} v\zeta(dv) = 1$ *is the degenerate probability distribution* $\zeta(\cdot) = \delta_1(\cdot)$;

(ii) *If* $P\{\sum_{j=1}^N B_j = 1\} < 1$ *and if* $a(s) < 1$ *for some* $s > 1$, *then equation* (54) *has a unique solution* ζ *with* $\int_{\mathbb{R}^+} \zeta(dv) = \int_{\mathbb{R}^+} v\zeta(dv) = 1$. *Moreover* ζ *is non-degenerate and, for any* $p > 1$, $\int_{\mathbb{R}^+} v^p \zeta(dv) < +\infty$ *if and only if* $a(p) < 1$.

Proof of Proposition 2.1. Equation (13) is of type (55) with $B_i = A_i^\gamma \Theta^{\mathcal{S}(\gamma)}$, Θ being an uniform random variable on $[0,1]$ independent from $(A_1, \dots, A_N)$. Hence, in this case, the function a defined in (56) becomes

$$a(s) := a_\gamma(s) = \frac{\mathcal{S}(\gamma s) + 1}{\mathcal{S}(\gamma)s + 1}.$$

Clearly $a_\gamma(1) = 1$. Now, since $\delta > \gamma$ and $\mathcal{S}(\gamma)/\gamma = \mu(\gamma) > \mu(\delta) \geq -1/\delta$, it is easy to see that the convex function $q \mapsto a_\gamma(q/\gamma)$ is equal to 1 in $q = \gamma$ and strictly smaller than 1 in $q = \delta$. Since $\mu(q) - \mu(\gamma) = 0$ if and only if $a_\gamma(q/\gamma) = 1$ it follows that equation $\mu(q) - \mu(\gamma) = 0$ has at most one solution $q_\gamma^* \neq \gamma$. This proves (ii). Noticing that $\delta/\gamma > 1$ and $a_\gamma(\delta/\gamma) < 1$, by Proposition 5.4 (i) follows. Since Θ and A are independent, $\Theta^{\mathcal{S}(\gamma)} \sum_{i=1}^N A_i^\gamma = 1$ almost surely if and only if $\sum_{i=1}^N A_i^\gamma = 1$ almost surely. Hence, by (ii) of Proposition 5.4 $\zeta_{\infty,\gamma}$ is degenerate if and only if $\sum_{i=1}^N A_i^\gamma = 1$ almost surely. Finally, using that a_γ is convex, $a_\gamma(1) = 1$ and $a_\gamma(\delta/\gamma) < 1$, it follows that $a_\gamma(p/\gamma) < 1$ if and only if $p < q_\gamma^*$. Again by (ii) of Proposition 5.4, whenever $\zeta_{\infty,\gamma}$ is non-degenerate, $\int_{\mathbb{R}^+} v^{\frac{p}{\gamma}} \zeta_{\infty,\gamma}(dv) < +\infty$ if and only $p < q_\gamma^*$. □

Proof of Theorem 2.2. Proposition 4.2 yields that $\mathbb{E}[\tilde{M}_\infty(\gamma)] = 1$, since $\mu(\gamma) > \mu(\delta)$ for $\delta > \gamma$. An easy computation shows that $\mathbb{E}[c_\gamma^\gamma Z^{\frac{\mu(\gamma)\gamma}{N-1}}] = 1$ and then $\mathbb{E}[c_\gamma^\gamma Z^{\frac{\mu(\gamma)\gamma}{N-1}} \tilde{M}_\infty(\gamma)] = 1$. Hence, recalling that $v_{\infty,\gamma}$ is the characteristic function of $c_\gamma^\gamma Z^{\frac{\mu(\gamma)\gamma}{N-1}} \tilde{M}_\infty(\gamma)$, by (51) of Proposition 5.3 and (i) of Proposition 2.1 the law of $c_\gamma^\gamma Z^{\frac{\mu(\gamma)\gamma}{N-1}} \tilde{M}_\infty(\gamma)$ is equal to $\zeta_{\infty,\gamma}$. At this stage (15) follows by (39). Moreover, from (52) of Proposition 5.3, $w_{\infty,\gamma}$ is a solution of equation (6) for $\mu^* = \mu(\gamma)$. The proof of (i) is completed.

In order to prove (ii) let us observe that, from the properties of γ-stable distributions, it follows that $\int_{\mathbb{R}^+} v^p \rho_{\infty,\gamma}(dv) < +\infty$ if and only if $p < \gamma$ and $\int_{\mathbb{R}^+} v^{\frac{p}{\gamma}} \zeta_{\infty,\gamma}(dv) < +\infty$, but for $p < \gamma$,

$$\int_{\mathbb{R}^+} v^{\frac{p}{\gamma}} \zeta_{\infty,\gamma}(dv) \leq \Big(\int_{\mathbb{R}^+} v \zeta_{\infty,\gamma}(dv) \Big)^{\frac{p}{\gamma}} = 1.$$

It remains to show that $\rho_{\infty,\gamma}$ is a γ-stable distribution if and only if $\sum_{i=1}^N A_i^\gamma = 1$ almost surely. This follows from (iii) of Proposition 2.1 and the fact that

$$e^{-|\xi|^\gamma k_1 (1 - i\eta_1 \tan(\pi\gamma/2) \operatorname{sign} \xi)} = \int_{\mathbb{R}^+} e^{-|\xi|^\gamma z k_0 (1 - i\eta_0 \tan(\pi\gamma/2) \operatorname{sign} \xi)} \zeta_{\infty,\gamma}(dz) \tag{57}$$

if and only if $k_1 = k_0$, $\eta_1 = \eta_0$ and $\zeta_{\infty,\gamma} = \delta_1$. Let us prove the last claim. Write (57) for $\xi > 0$ with $\xi^\gamma = x$, and differentiate the resulting identity with respect to x to obtain

$$\begin{aligned} &-k_1(1 - i\eta_1 \tan(\pi\gamma/2)) e^{-x k_1 (1 - i\eta_1 \tan(\pi\gamma/2))} \\ &\qquad = \int_{\mathbb{R}^+} k_0 z (1 - i\eta_0 \tan(\pi\gamma/2)) e^{-x k_0 (1 - i\eta_0 \tan(\pi\gamma/2))} \zeta_{\infty,\gamma}(dz). \end{aligned} \tag{58}$$

Taking the limit for $x \downarrow 0$, recalling that $\int_{\mathbb{R}^+} z\zeta_{\infty,\gamma}(dz) = 1$, by dominated convergence one gets

$$k_1(1 - i\eta_1 \tan(\pi\gamma/2)) = k_0(1 - i\eta_0 \tan(\pi\gamma/2))$$

and hence $k_1 = k_0$ and $\eta_0 = \eta_1$. At this stage it suffices to recall that a scale mixture of Stable laws is an identifiable family of distributions. See, for example, [27].

Analogously, (iii) and (iv) follow from (51) of Proposition 5.3 and from (iii) of Proposition 2.1.

□

Proof of Theorem 2.3. Recall that $\mu(\delta) < \mu(\gamma)$ for $\delta > \gamma$, hence by Proposition 4.2 yields that $\tilde{M}_\infty(\gamma) = 0$ and this completes the proof. □

Proof of Theorem 2.4. We shall assume that $l_\delta(X_0, V_\infty) < +\infty$, since otherwise the claim is trivial. Then, there exists an optimal pair (X^*, Y^*) realizing the infimum in the definition of the Wasserstein distance,

$$\Delta := l_\delta^{\max(\delta,1)}(X_0, V_\infty) = l_\delta^{\max(\delta,1)}(X^*, Y^*) = \mathbb{E}|X^* - Y^*|^\delta. \tag{59}$$

Let $(X_v^*, Y_v^*)_{v\in\mathbb{U}}$ be a sequence of independent and identically distributed random variables with the same law of (X^*, Y^*), which are further independent of $(\nu_t)_{t\geq 0}$, $(T_n)_{n\geq 1}$, $(A(v))_{v\in\mathbb{U}}$. By Proposition 3.2 it follows that $\sum_{j=1}^{f_{\nu_t}} X_{j,\nu_t}^* \beta_{j,\nu_t}$ has the same law of V_t, where $X_{j,n}^* = X_{L_{j,n}}^*$ and $L_{j,n}$ is defined at the end of Section 3. Moreover, since the characteristic function of V_∞ is a solution of (6) with $\mu^* = \mu(\gamma)$, as already noted in the Introduction, the characteristic function of $e^{\mu(\gamma)t}V_\infty$ is a solution of (1) with $\phi_0 = w_{\infty,\gamma}$. Hence, applying once again Proposition 3.2, we get that $e^{\mu(\gamma)t}V_\infty$ has the same law of $\sum_{j=1}^{f_{\nu_t}} Y_{j,\nu_t}^* \beta_{j,\nu_t}$, where $Y_{j,n}^* = Y_{L_{j,n}}^*$ For the sake of simplicity write (X_j^*, Y_j^*) in place of $(X_{j,n}^*, Y_{j,n}^*)$. We can write

$$\begin{aligned}
l_\delta^{\max(\delta,1)}(e^{-\mu(\gamma)t}V_t, V_\infty) &= l_\delta^{\max(\delta,1)}(e^{-\mu(\gamma)t}V_t, e^{-\mu(\gamma)t}e^{\mu(\gamma)t}V_\infty) \\
&= e^{-\delta\mu(\gamma)t} l_\delta^{\max(\delta,1)}(V_t, e^{\mu(\gamma)t}V_\infty) \\
&\leq e^{-\delta\mu(\gamma)t} \sum_{n\geq 0} \zeta(t,n) \mathbb{E}\Big[\Big|\sum_{j=1}^{f_n} X_j^*\beta_{j,n} - \sum_{j=1}^{f_n} Y_j^*\beta_{j,n}\Big|^\delta\Big] \\
&= e^{-\delta\mu(\gamma)t} \sum_{n\geq 0} \zeta(t,n) \mathbb{E}\Big[\mathbb{E}\Big[\Big|\sum_{j=1}^{f_n}(X_j^* - Y_j^*)\beta_{j,n}\Big|^\delta \Big| \mathcal{G}_n\Big]\Big],
\end{aligned}$$

where $\zeta(t,n)$ is the density of ν_t (see (28)) and $\mathcal{G}_n = \sigma(A(v)_{v\in T_n}, T_1, \ldots, T_n)$. Now, if $0 < \gamma < \delta \leq 1$, then Minkowski's inequality yields

$$\mathbb{E}\Big[\mathbb{E}\Big[\Big|\sum_{j=1}^{f_n}(X_j^* - Y_j^*)\beta_{j,n}\Big|^\delta \Big| \mathcal{G}_n\Big]\Big] \leq \mathbb{E}\Big[\mathbb{E}\Big[\sum_{j=1}^{f_n} \beta_{j,n}^\delta |X_j^* - Y_j^*|^\delta \Big| \mathcal{G}_n\Big]\Big] = \mathbb{E}\Big[\sum_{j=1}^{f_n} \beta_{j,n}^\delta\Big]\Delta; \tag{60}$$

where Δ is defined in (59). We now want to prove a similar inequality for $1 \leq \gamma < \delta \leq 2$. First of all we need to observe that in addition to $\mathbb{E}|X_j^* - Y_j^*|^\delta = l_\delta^\delta(X_0, V_\infty) < +\infty$, we have also that $\mathbb{E}(X_j^* - Y_j^*) = 0$. If $\gamma \neq 1$ the claim follows since by hypothesis $\mathbb{E}(X_j^*) = \mathbb{E}(X_0) = 0$ and

$\mathbb{E}(Y_j^*) = \mathbb{E}(V_\infty) = 0$, thanks to the fact that V_∞ is a mixture of centered stable random variables of exponent $\gamma > 1$. When $\gamma = 1$ and (a) of $(\mathbf{H}_1)$ holds the proof of the claim is similar. When $\gamma = 1$ and (b) of $(\mathbf{H}_1)$ holds, the proof requires more care, since $\mathbb{E}|X_0| = \mathbb{E}|V_\infty| = +\infty$. Let $F_\infty(y)$ be the probability distribution function of V_∞, that is $F_\infty(x) = \int_{(-\infty,x]} \rho_{\infty,\gamma}(dy)$, and recall that as an optimal pair one can choose $(X^*, Y^*) = (F_0^{-1}(U), F_\infty^{-1}(U))$, U being a uniform random variable on $(0,1)$ and F_0^{-1} (F_∞^{-1}, respectively) is the quantile function of F_0 (F_∞, respectively), see, e.g., [25]. Note that $\mathbb{E}|X_j^* - Y_j^*|^\delta < +\infty$, which yields that $\mathbb{E}|X_j^* - Y_j^*| = \int_0^1 |F_0^{-1}(u) - F_\infty^{-1}(u)|du < +\infty$. Since F_0 and F_∞ are symmetric distribution functions it is easy to see that $F_0^{-1}(U) - F_\infty^{-1}(U)$ is a symmetric random variable and hence $\mathbb{E}(F_0^{-1}(U) - F_\infty^{-1}(U)) = \mathbb{E}(X_j^* - Y_j^*) = 0$. Summarizing, if $1 \le \gamma \le 2$, we have $\mathbb{E}(X_j^* - Y_j^*) = 0$ and $\mathbb{E}|X_j^* - Y_j^*|^\delta < +\infty$, hence we can apply the Bahr-Esseen inequality (see [28]) to obtain

$$\mathbb{E}\Big[\mathbb{E}\Big[\Big|\sum_{j=1}^{f_n}(X_j^* - Y_j^*)\beta_{j,n}\Big|^\delta \Big|\mathcal{G}_n\Big]\Big] \le \mathbb{E}\Big[\Big[2\sum_{j=1}^{f_n}\beta_{j,n}^\delta |X_j^* - Y_j^*|^\delta \Big|\mathcal{G}_n\Big] = 2\mathbb{E}\Big[\sum_{j=1}^{f_n}\beta_{j,n}^\delta\Big]\Delta. \tag{61}$$

Combining (60)-(61) with Proposition 4.1 we obtain

$$\begin{aligned} l_\delta^{\max(\delta,1)}(e^{-\mu(\gamma)}V_t, V_\infty) &\le c\Delta e^{-\delta\mu(\gamma)t}\sum_{n\ge 0}\zeta(t,n)\mathbb{E}\Big[\sum_{j=1}^{n}\beta_{j,n}^\delta\Big] \\ &= c\Delta e^{-\delta\mu(\gamma)t}\sum_{n\ge 0} e^{-t}(1-e^{-(N-1)t})^n \frac{(\frac{1}{N-1})_n}{n!}\frac{(\frac{\mathcal{S}(\delta)+1}{N-1})_n}{(\frac{1}{N-1})_n} \\ &= c\Delta e^{-\delta\mu(\gamma)t}\sum_{n\ge 0} e^{-t}(1-e^{-(N-1)t})^n \frac{(\frac{\mathcal{S}(\delta)+1}{N-1})_n}{n!} \end{aligned}$$

with $c = 1$ if $0 < \gamma < \delta \le 1$ and $c = 2$ if $1 \le \gamma < \delta \le 2$, and the thesis follows since

$$\sum_{n\ge 0}\frac{(r)_n}{n!}(1-q)^n = q^{-r}$$

for every $q \in (0,1)$ and $r > 0$. □

Proof of Lemma 2.5. The proof of this Lemma follows step by step the proof of Lemma 1 in [2]. By Lemma 9 in [2], if $\delta < \gamma/(1-\epsilon)$, it suffices to prove that the probability distribution function of V_∞, that is $F_\infty(x) = \int_{(-\infty,x]}\rho_{\infty,\gamma}(dy)$, satisfies (20)-(21) with the same constants c_0^+ and c_0^- as the initial condition F_0 (possibly after diminishing ϵ and enlarging K). The proof is based on the representation of F_∞ as a mixture of stable laws. More precisely, let G_γ be the distribution function whose Fourier-Stieltjes transform is $\hat{g}_\gamma$ as in (11), then

$$F_\infty(x) = \mathbb{E}\Big[G_\gamma\Big(\big(Y\big)^{-1/\gamma}x\Big)\Big],$$

where Y has distribution $\zeta_{\gamma,\infty}$, see Theorem 2.2. Since $\gamma < \delta < 2\gamma$, then there exists a finite constant $K > 0$ such that $|1 - c_0^+ x^{-\gamma} - G_\gamma(x)| \le Kx^{-\delta}$ for $x > 0$, and similarly for $x < 0$; see, e.g.

Sections 2.4 and 2.5 of [30]. Using that $\mathbb{E}[Y]=1$ and $C:=\mathbb{E}[Y^{\delta/\gamma}]<\infty$ (by (iii) of Proposition 2.1 since $\delta<q_\gamma^*$) it follows further that

$$\begin{aligned}\left|1-c_0^+x^{-\gamma}-F_\infty(x)\right| &\leq \mathbb{E}\big[\big|1-c_0^+\big((Y)^{-1/\gamma}x\big)^{-\gamma}-G_\gamma(Y^{-1/\gamma}x)\big|\big]\\ &\leq \mathbb{E}\big[K(Y)^{\delta/\gamma}x^{-\delta}\big]=CKx^{-\delta}.\end{aligned}$$

This proves (20) for F_∞, with $\epsilon=\delta-\gamma$ and $K'=CK$. A similar argument proves (21). □

Proof of Theorem 2.6. The proof follows the same line of the proof of Theorem 2.4. Assume that $\mathcal{Z}_\delta(X_0,V_\infty)<+\infty$, since otherwise the claim is trivial. Consider two sequences of independent and identically distributed random variable $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$, X_j with common distribution function F_0 and Y_j with the same law of V_∞. In addition assume that $(X_j)_{j\geq 1}$,$(Y_j)_{j\geq 1}$, $(\nu_t)_{t\geq 0}$ and $(\beta_{j,n})_{j,n}$ are stochastically independent. Recall that, as noted in the proof of Theorem 2.4, $e^{\mu(\gamma)t}V_\infty$ has the same law of $\sum_{j=1}^{f_{\nu_t}} Y_j\beta_{j,\nu_t}$. First of all it is clear, by the definition of $\mathcal{Z}_\delta$, that

$$\begin{aligned}\mathcal{Z}_\delta(e^{-\mu(\gamma)t}V_t,V_\infty) &= \mathcal{Z}_\delta\Big(e^{-\mu(\gamma)t}\sum_{j=1}^{f_{\nu_t}}X_j\beta_{j,\nu_t},e^{-\mu(\gamma)t}\sum_{j=1}^{f_{\nu_t}}Y_j\beta_{j,\nu_t}\Big)\\ &\leq \sum_{n\geq 0}\zeta(t,n)\mathcal{Z}_\delta\Big(e^{-\mu(\gamma)t}\sum_{j=1}^{f_n}X_j\beta_{j,n},e^{-\mu(\gamma)t}\sum_{j=1}^{f_n}Y_j\beta_{j,n}\Big).\end{aligned}$$

An important property of the Zolotarev's metric $\mathcal{Z}_\delta$ is that it is ideal of order δ, that is

$$\mathcal{Z}_\delta(cX,cY)=c^\delta\mathcal{Z}_\delta(X,Y)$$

(see, e.g., Theorem 1.4.2 in [31]), which yields that

$$\mathcal{Z}_\delta\Big(e^{-\mu(\gamma)t}V_t,V_\infty\Big)\leq\sum_{n\geq 0}\zeta(t,n)e^{-\delta\mu(\gamma)t}\mathcal{Z}_\delta\Big(\sum_{j=1}^{f_n}X_j\beta_{j,n},\sum_{j=1}^{f_n}Y_j\beta_{j,n}\Big).$$

Now, by Proposition 1 in [26],

$$\mathcal{Z}_\delta\Big(\sum_{j=1}^{f_n}X_j\beta_{j,n},\sum_{j=1}^{f_n}Y_j\beta_{j,n}\Big)\leq\mathbb{E}\Big[\sum_{j=1}^{f_n}\beta_{j,n}^\delta\Big]\mathcal{Z}_\delta(X_0,V_\infty).$$

In conclusion we get

$$\mathcal{Z}_\delta\Big(e^{-\mu(\gamma)t}V_t,V_\infty\Big)\leq\mathcal{Z}_\delta(X_0,V_\infty)e^{-\delta\mu(\gamma)t}\sum_{n\geq 0}\zeta(t,n)\mathbb{E}\Big[\sum_{j=1}^{f_n}\beta_{j,n}^\delta\Big].$$

At this stage the first part of the thesis follows exactly as in the last part of the proof of Theorem 2.4. It remains to show that, if $\gamma=2$, $\delta\leq 3$ and $\mathbb{E}|X_0|^\delta<+\infty$, then

$$\mathcal{Z}_\delta(X_0,V_\infty)\leq\frac{1}{\Gamma(1+\delta)}\Big[\mathbb{E}|X_0|^\delta+\mathbb{E}|V_\infty|^\delta\Big]$$

which, by Theorem 2.2 (iii) is finite. To prove the last inequality recall that, given two random variable X and Y, if $2<\delta\leq 3$, then

$$\mathcal{Z}_\delta(X,Y)\leq\frac{1}{\Gamma(1+\delta)}\Big[\mathbb{E}|X|^\delta+\mathbb{E}|Y|^\delta\Big]$$

provided that $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbb{E}[X^2] = \mathbb{E}[Y^2]$, see Theorem 1.5.7 in [31]. In our case by Theorem 2.2 (i)-(iii), $\mathbb{E}[X_0] = \mathbb{E}[V_\infty] = 0$, $\mathbb{E}[X_0^2] = \sigma_0^2$, $\mathbb{E}[V_\infty^2] = \sigma_0^2 \int_{\mathbb{R}^+} z\zeta_{\infty,2}(dz) = \sigma_0^2$. □

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